COACTIONS ON HOCHSCHILD HOMOLOGY OF HOPF-GALOIS EXTENSIONS AND THEIR COINVARIANTS

A. MAKHLOUF AND D. ŞTEFAN

ABSTRACT. Let $\mathcal{B} \subseteq \mathcal{A}$ be an $\mathcal{H}$-Galois extension, where $\mathcal{H}$ is a Hopf algebra over a field $\mathbb{K}$. If M is a Hopf bimodule then $\mathrm{HH}_*(\mathcal{A}, M)$, the Hochschild homology of $\mathcal{A}$ with coefficients in M, is a right comodule over the coalgebra $\mathcal{C}_\mathcal{H} = \mathcal{H}/[\mathcal{H}, \mathcal{H}]$. Given an injective left $\mathcal{C}_\mathcal{H}$-comodule V, our aim is to understand the relationship between $\mathrm{HH}_*(\mathcal{A}, M) \square_{\mathcal{C}_\mathcal{H}} V$ and $\mathrm{HH}_*(\mathcal{B}, M\square_{\mathcal{C}_\mathcal{H}} V)$. The roots of this problem can be found in [Lo2], where $\mathrm{HH}_*(\mathcal{A}, \mathcal{A})^G$ and $\mathrm{HH}_*(\mathcal{B}, \mathcal{B})$ are shown to be isomorphic for any centrally G-Galois extension. To approach the above mentioned problem, in the case when $\mathcal{A}$ is a faithfully flat $\mathcal{B}$-module and $\mathcal{H}$ satisfies some technical conditions, we construct a spectral sequence

$$\mathrm{Tor}_p^{\mathcal{R}_\mathcal{H}} \left(\mathbb{K}, \mathrm{HH}_q(\mathcal{B}, M\square_{\mathcal{C}_\mathcal{H}} V) \right) \Longrightarrow \mathrm{HH}_{p+q}(\mathcal{A}, M) \square_{\mathcal{C}_\mathcal{H}} V,$$

where $\mathcal{R}_\mathcal{H}$ denotes the subalgebra of cocommutative elements in $\mathcal{H}$. We also find conditions on $\mathcal{H}$ such that the edge maps of the above spectral sequence yield isomorphisms

$$\mathbb{K} \otimes_{\mathcal{R}_\mathcal{H}} \mathrm{HH}_*(\mathcal{B}, M\square_{\mathcal{C}_\mathcal{H}} V) \cong \mathrm{HH}_*(\mathcal{A}, M)\square_{\mathcal{C}_\mathcal{H}} V.$$

In the last part of the paper we define centrally Hopf-Galois extensions and we show that for such an extension $\mathcal{B} \subseteq \mathcal{A}$, the $\mathcal{R}_\mathcal{H}$-action on $\mathrm{HH}_*(\mathcal{B}, M\square_{\mathcal{C}_\mathcal{H}} V)$ is trivial. As an application, we compute the subspace of $\mathcal{H}$-coinvariant elements in $\mathrm{HH}_*(\mathcal{A}, M)$. A similar result is derived for $\mathrm{HC}_*(\mathcal{A})$, the cyclic homology of $\mathcal{A}$.

INTRODUCTION

Chase and Sweedler [CS], more than 35 years ago, defined a special case of Hopf-Galois extensions, similar to the theory of Galois group actions on commutative rings that had been developed by Chase, Harrison and Rosenberg [CHR]. The general definition, for arbitrary Hopf algebras, is due to Takeuchi and Kreimer [KT]. Besides Galois group actions that we already mentioned, strongly graded algebras are Hopf-Galois extensions. Other examples come from the theory of affine quotients and of enveloping algebras of Lie algebras.

By definition, an $\mathcal{H}$-Galois extension is given by an algebra map $\rho_\mathcal{A} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$ that defines a coaction of a Hopf algebra $\mathcal{H}$ on the algebra $\mathcal{A}$ such that the map

$$\beta_\mathcal{A} : \mathcal{A} \otimes_\mathcal{B} \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}, \quad \beta_A = (m_\mathcal{A} \otimes \mathcal{H}) \circ (\mathcal{A} \otimes_\mathcal{B} \rho_\mathcal{A})$$

is bijective. Here $m_\mathcal{A} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denotes the multiplication map in $\mathcal{A}$ and $\mathcal{B}$ is the subalgebra of coinvariant elements, i.e. of all $a \in A$ such that $\rho_\mathcal{A}(a) = a \otimes 1$. For the remaining part of the introduction we fix an $\mathcal{H}$-Galois extension $\mathcal{B} \subseteq \mathcal{A}$. If there is no danger of confusion, we shall say that $\mathcal{B} \subseteq \mathcal{A}$ is a Hopf-Galois extension.

For an $\mathcal{A}$-bimodule M, let $\mathrm{HH}_*(\mathcal{A}, M)$ denote Hochschild homology of $\mathcal{A}$ with coefficients in M. Since $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, the Hochschild homology of $\mathcal{B}$ with coefficients in M also makes sense. A striking feature of $\mathrm{HH}_*(\mathcal{B}, M)$ is that $\mathcal{H}$ acts on these linear spaces via the Ulbrich-Miyashita action, cf. [Ş1]. By taking M to be a Hopf bimodule, more structure can be defined not only on $\mathrm{HH}_*(\mathcal{B}, M)$ but on $\mathrm{HH}_*(\mathcal{A}, M)$ too. Recall that M is a Hopf bimodule if M is an $\mathcal{A}$-bimodule and an $\mathcal{H}$-comodule such that the maps that define the module structures are $\mathcal{H}$-colinear. By definition of Hopf modules, $\mathrm{HH}_0(\mathcal{B}, M)$ is a quotient $\mathcal{H}$-comodule of M. This structure can be extended to an $\mathcal{H}$-coaction on $\mathrm{HH}_n(\mathcal{B}, M)$, for every n. On the other hand, $\mathrm{HH}_0(\mathcal{A}, M)$ is not an $\mathcal{H}$-comodule, in general. Nevertheless, the quotient coalgebra $\mathcal{C}_\mathcal{H} := \mathcal{H}/[\mathcal{H}, \mathcal{H}]$ coacts on $\mathrm{HH}_*(\mathcal{A}, M)$, where $[\mathcal{H}, \mathcal{H}]$ denotes the subspace spanned by all commutators in $\mathcal{H}$, see [Ş2].

These actions and coactions played an important role in the study of Hochschild (co)homology of Hopf-Galois extensions, having deep application in the field. Let us briefly discuss some of them, that are related to the present work. First, in degree zero, the coinvariants of Ulbrich-Miyashita action

$$\mathrm{HH}_0(\mathcal{B}, M)_\mathcal{H} := \mathbb{K} \otimes_\mathcal{H} \mathrm{HH}_0(\mathcal{B}, M)$$

equal $\mathrm{HH}_0(\mathcal{A}, M)$. This identification is one of the main ingredients that are used in [Ş1] to prove the existence of the spectral sequence

$$\mathrm{Tor}_p^\mathcal{H}(\mathbb{K}, \mathrm{HH}_q(\mathcal{B}, M)) \implies \mathrm{HH}_*(\mathcal{A}, M). \tag{1}$$

It generalizes, in an unifying way, Lyndon-Hochschild-Serre spectral sequence for group homology [We, p. 195], Hochschild-Serre spectral sequence for Lie algebra homology [We, p. 232] and Lorenz spectral sequence for strongly graded algebras [Lo1]. If $\mathcal{H}$ is semisimple then (1) collapses and yields the isomorphisms

$$\mathrm{HH}_n(\mathcal{B}, M)^\mathcal{H} \cong \mathrm{HH}_n(\mathcal{B}, M)_\mathcal{H} \cong \mathrm{HH}_n(\mathcal{A}, M),$$

where $\mathrm{HH}_n(\mathcal{B}, M)^\mathcal{H}$ denotes the space of $\mathcal{H}$-invariant Hochschild homology classes. Similar isomorphisms in Hochschild cohomology were proved in the same way and used, for example, to investigate algebraic deformations arising from orbifolds with discrete torsion [CGW], to characterize deformations of certain bialgebras [MW] and to study the G-structure on the cohomology of a Hopf algebra [FS].

In [Ş2], for a subcoalgebra C of $\mathcal{C}_\mathcal{H}$, a new homology theory $\mathrm{HH}_*^C(\mathcal{A}, -)$ with coefficients in the category of Hopf bimodules is defined. In the case when C is injective as a left $\mathcal{C}_\mathcal{H}$-comodule we have

$$\mathrm{HH}_*^C(\mathcal{A}, M) \cong \mathrm{HH}_*(\mathcal{A}, M) \square_{\mathcal{C}_\mathcal{H}} C,$$

for every Hopf bimodule M (for the definition of the cotensor product $\square_{\mathcal{C}_\mathcal{H}}$ see the preliminaries of this paper). Thus, $\mathrm{HH}_*^C(\mathcal{A}, M)$ may be regarded as a sort of C-coinvariant part of $\mathrm{HH}_*(\mathcal{A}, M)$. The main result in loc. cit. is the spectral sequence

$$\mathrm{Tor}_p^\mathcal{H}(\mathbb{K}, \mathrm{HH}_q(\mathcal{B}, M \square_{\mathcal{C}_\mathcal{H}} C)) \implies \mathrm{HH}_*^C(\mathcal{A}, M), \tag{2}$$

that exists for every Hopf bimodule M, provided that $\mathcal{H}$ is cocommutative and C is injective as a left $\mathcal{C}_\mathcal{H}$-comodule.

Let G be a group and let $\mathbb{K}$ be a field. If $\mathcal{H}$ is the group algebra $\mathbb{K}G$, then $\mathcal{C}_{\mathbb{K}G} := \bigoplus_{\sigma \in T(G)} C_\sigma$, where $T(G)$ denotes the set of conjugacy classes in G and C_σ is

a subcoalgebra of dimension one, for every $\sigma \in T(G)$. Hence, in this particular case, the homogeneous components $\mathrm{HH}_*^\sigma(\mathcal{A}, M) := \mathrm{HH}_*^{C_\sigma}(\mathcal{A}, M)$ completely determine Hochschild homology of $\mathcal{A}$ with coefficients in M, cf. [Lo1, §2]. For strongly graded algebras (i.e. $\mathbb{K}G$-Galois extensions) and $C = C_\sigma$, the spectral sequence (2) is due to Lorenz [Lo1]. On the other hand, Burghelea and Nistor defined and studied similar homogeneous components of Hochschild and cyclic cohomology of group algebras and crossed products in [Bu] and [Ni], respectively.

We have already remarked that, for an arbitrary $\mathcal{H}$-Galois extension $\mathcal{B} \subseteq \mathcal{A}$ and every Hopf bimodule M, Hochschild homology $\mathrm{HH}_*(\mathcal{B}, M)$ is a right $\mathcal{H}$-comodule and a left $\mathcal{H}$-module. In particular, $\mathcal{H}$ acts and coacts on $\mathcal{A}_{\mathcal{B}} := \mathrm{HH}_0(\mathcal{B}, \mathcal{A})$. Notably, with respect to these structures, $\mathcal{A}_{\mathcal{B}}$ is a stable-anti-Yetter-Drinfeld $\mathcal{H}$-module (SAYD $\mathcal{H}$-module, for short). These modules were independently discovered in [JŞ] and [HKRS], and they can be thought of as coefficients for Hopf-cyclic homology. In [JŞ] the authors showed that Hopf-cyclic homology of $\mathcal{H}$ with coefficients in $\mathcal{A}_{\mathcal{B}}$ equals relative cyclic homology $\mathrm{HC}_*(\mathcal{A}/\mathcal{B})$. This identification is then used to compute cyclic homology of a strongly G-graded algebra with separable component of degree 1 (e.g. group algebras and quantum tori). It is worthwhile to mention that cyclic homology of a groupoid was computed in [BŞ], using the theory of (generalized) SAYD modules.

Let G be a finite group of automorphisms of an algebra $\mathcal{A}$ over a field $\mathbb{K}$ and let $\mathcal{A}^G$ denote the ring of G-invariants in $\mathcal{A}$. Since G is finite, the dual vector space $(\mathbb{K}G)^*$ has a canonical structure of Hopf algebra and $\mathcal{A}$ is a $(\mathbb{K}G)^*$-comodule algebra. Clearly, the coinvariant subalgebra with respect to this coaction equals $\mathcal{A}^G$. It is well-known that $\mathcal{A}^G \subseteq \mathcal{A}$ is $(\mathbb{K}G)^*$-Galois if and only if this extension is Galois in the sense of [CHR]. The center $\mathcal{Z}$ of $\mathcal{A}$ is G-invariant. Following [Lo2], we say that $\mathcal{A}^G \subseteq \mathcal{A}$ is centrally Galois if $\mathcal{Z}^G \subseteq \mathcal{Z}$ is $(\mathbb{K}G)^*$-Galois.

The Galois group G acts, of course both on Hochschild homology $\mathrm{HH}_*(\mathcal{A}, \mathcal{A})$ and cyclic homology $\mathrm{HC}_*(\mathcal{A})$. To simplify the notation, we shall write $\mathrm{HH}_*(\mathcal{A})$ for $\mathrm{HH}_*(\mathcal{A}, \mathcal{A})$. By [Lo2, §6], for a centrally Galois extension $\mathcal{A}^G \subseteq \mathcal{A}$,

$$\mathrm{HH}_*(\mathcal{A})^G \cong \mathrm{HH}_*(\mathcal{A}^G) \tag{3}$$

and a similar isomorphism exists in cyclic homology, provided that the order of G is invertible in $\mathbb{K}$.

Since $(\mathbb{K}G)^*$ is commutative, the coalgebras $\mathcal{C}_{(\mathbb{K}G)^*}$ and $(\mathbb{K}G)^*$ are equal. The category of left $\mathbb{K}G$-modules is isomorphic to the category of right $(\mathbb{K}G)^*$-comodules and through this identification $X^G \cong X\square_{(\mathbb{K}G)^*}\mathbb{K}$. In particular,

$$\mathrm{HH}_*(\mathcal{A})^G \cong \mathrm{HH}_*(\mathcal{A})\square_{(\mathbb{K}G)^*}\mathbb{K}.$$

This isomorphism suggests that the main result in [Lo2] might be approached in the spirit of [Ş2], i.e. using the theory of Hopf-Galois extensions and an appropriate spectral sequence that converges to $\mathrm{HH}_*(\mathcal{A})\square_{(\mathbb{K}G)^*}\mathbb{K}$. Since in general $(\mathbb{K}G)^*$ is not cocommutative, the spectral sequence in (2) cannot be used directly. The main obstruction to extend it for a not necessarily cocommutative Hopf algebra $\mathcal{H}$, is the fact that Ulbrich-Miyashita action does not induce an $\mathcal{H}$-action on $\mathrm{HH}_*(\mathcal{B}, M\square_{\mathcal{C}_\mathcal{H}}C)$.

To overcome this difficulty we define

$$\mathcal{R}_{\mathcal{H}} := \ker\left(\Delta - \tau \circ \Delta\right),$$

where $\tau : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ denotes the usual flip map. Since Δ and $\tau \circ \Delta$ are morphisms of algebras, $\mathcal{R}_{\mathcal{H}}$ is a subalgebra in $\mathcal{H}$. Moreover, if $\mathcal{A}$ is faithfully flat as a left (or right) $\mathcal{B}$-module and the antipode of $\mathcal{H}$ is an involution, then we prove that $\mathrm{HH}_*(\mathcal{B}, M\square_{\mathcal{C}_{\mathcal{H}}}V)$ are left $\mathcal{R}_{\mathcal{H}}$-modules, for every injective left $\mathcal{C}_{\mathcal{H}}$-comodule V; see Proposition 2.14. Thus, under the above assumptions, for every pair (p,q) of natural numbers, it makes sense to define the vector spaces

$$E_{p,q}^2 := \mathrm{Tor}_p^{\mathcal{R}_{\mathcal{H}}}\left(\mathbb{K}, \mathrm{HH}_q(\mathcal{B}, M\square_{\mathcal{C}_{\mathcal{H}}}V)\right). \tag{4}$$

Furthermore, in Theorem 2.23 we prove that there is a spectral sequence that has $E_{p,q}^2$ in the (p,q)-spot of the second page and converges to $\mathrm{HH}_*(\mathcal{A}, M)\square_{\mathcal{H}}V$. This result follows as an application of Proposition 2.3, where we indicate a new form of Grothendieck's spectral sequence. It also relies on several properties of the Ulbrich-Miyashita action that are proved in the first part of paper. Here, we just mention equation (21) that plays a key role, as it explains the relationship between the module and comodule structures on $\mathrm{HH}_*(\mathcal{B}, M)$. More precisely, using the terminology from [HKRS, JŞ], relation (21) means that Hochschild homology of $\mathcal{B}$ with coefficients in M is an SAYD $\mathcal{H}$-module.

The most restrictive conditions that we impose in Theorem 2.23 are the relations

$$\mathcal{R}_{\mathcal{H}}^+\mathcal{H} = \mathcal{H}^+ \quad \text{and} \quad \mathrm{Tor}_n^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, \mathcal{H}) = 0, \tag{5}$$

for every $n > 0$, where $\mathcal{H}^+$ is the kernel of the counit and $\mathcal{R}_{\mathcal{H}}^+ := \mathcal{R}_{\mathcal{H}} \bigcap \mathcal{H}^+$. The second relation in (5) is easier to handle. For example, if $\mathcal{H}$ is semisimple and cosemisimple over a field of characteristic zero we show that $\mathcal{R}_{\mathcal{H}}$ is semisimple and $\mathcal{C}_{\mathcal{H}}$ is cosemisimple, cf. Proposition 2.24. The proof of this result is based on the identification $\mathcal{R}_{\mathcal{H}^*} \cong \mathbb{K}\otimes_{\mathbb{Q}}C_{\mathbb{Q}}(\mathcal{H})$, where $C_{\mathbb{Q}}(\mathcal{H})$ denotes the character algebra of $\mathcal{H}$, and on the fact that $C_{\mathbb{Q}}(\mathcal{H})$ is semisimple if $\mathcal{H}$ is so. Thus in this case the second relation in (5) holds true. Obviously both relations in (5) are satisfied if $\mathcal{H}$ is cocommutative. Notably, by Proposition 2.31, they are also verified if $\mathcal{H}$ is semisimple and commutative. The other two assumptions in Theorem 2.23 are not very strong. The antipode of $\mathcal{H}$ is involutive for commutative and cocommutative Hopf algebras. In characteristic zero, by a result of Larson and Radford, the antipode of a finite-dimensional Hopf algebra is an involution if and only if the Hopf algebra is semisimple and cosemisimple. Faithfully flat Hopf-Galois extensions are characterized in [SS, Theorem 4.10]. In view of this theorem, if the antipode of $\mathcal{H}$ is bijective, then an $\mathcal{H}$-Galois extension $\mathcal{B} \subseteq \mathcal{A}$ is faithfully flat if and only if $\mathcal{A}$ is injective as an $\mathcal{H}$-comodule. Hence, every $\mathcal{H}$-Galois extension is faithfully flat if $\mathcal{H}$ is cosemisimple and its antipode is bijective.

If the algebra $\mathcal{R}_{\mathcal{H}}$ is semisimple then the spectral sequence in Theorem 2.23 collapses. We have already noticed that $\mathcal{R}_{\mathcal{H}}$ is semisimple if $\mathcal{H}$ is either semisimple and cosemisimple over a field of characteristic zero, or commutative and semisimple. In these situations, the edge maps induce isomorphisms

$$\mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} \mathrm{HH}_n(\mathcal{B}, M\square_{\mathcal{H}}V) \cong \mathrm{HH}_n(\mathcal{A}, M)\square_{\mathcal{H}}V. \tag{6}$$

By specializing the isomorphism in (6) to the case $\mathcal{H} := (\mathbb{K}G)^*$, we deduce the isomorphism in Corollary 2.35 that can be thought of as a generalization of (3).

Our result also explains why the isomorphism (3) works for centrally Galois extensions but not for arbitrary ones. Namely, the action of $\mathcal{R}_{(\mathbb{K}G)^*}$ on $\mathrm{HH}_n(\mathcal{B})$ is trivial for centrally Galois extensions, but not in general. In the last part of the paper we show that a similar result holds for Hopf-Galois extensions. Let $\mathcal{B} \subseteq \mathcal{A}$ be an $\mathcal{H}$-comodule algebra, where $\mathcal{H}$ is a commutative Hopf algebra. If the center $\mathcal{Z}$ of $\mathcal{A}$ is an $\mathcal{H}$-subcomodule of $\mathcal{A}$ and $\mathcal{Z} \bigcap \mathcal{B} \subseteq \mathcal{Z}$ is a faithfully flat $\mathcal{H}$-Galois extension, then we say that $\mathcal{B} \subseteq \mathcal{A}$ is centrally $\mathcal{H}$-Galois. We fix such an extension $\mathcal{B} \subseteq \mathcal{A}$. In view of Proposition 3.5, the $\mathcal{R}_\mathcal{H}$-action on $\mathrm{HH}_*(\mathcal{B}, M \square_\mathcal{H} V)$ is trivial. If $\mathcal{H}$ is finite-dimensional and $\dim \mathcal{H}$ is not zero in $\mathbb{K}$, then by Theorem 3.6

$$\mathrm{HH}_*(\mathcal{A}, M)\square_\mathcal{H} V \simeq \mathrm{HH}_*(\mathcal{B}, M\square_\mathcal{H} V),$$

for every Hopf bimodule M which is symmetric as a $\mathcal{Z}$-bimodule and every left $\mathcal{H}$-comodule V. Assuming that $\mathcal{H} := (\mathbb{K}G)^*$ and that the order of G is not zero in $\mathbb{K}$, and taking $M := \mathcal{A}$ and $V := \mathbb{K}$ in the above isomorphism, we obtain the isomorphism in (3), cf. Corollary 3.9. Another application of Theorem 3.6 is given in Corollary 3.7.

Our approach has also the advantage that one can easily recover $\mathrm{HH}_*(\mathcal{A}, M)$ from $\mathrm{HH}_*(\mathcal{B}, M^{\mathrm{co}\mathcal{H}})$. More precisely, in Theorem 3.11, we prove the following isomorphism of $\mathcal{Z}$-modules and $\mathcal{H}$-comodules

$$\mathrm{HH}_*(\mathcal{A}, M) \simeq \mathcal{Z} \otimes_{\mathcal{Z} \bigcap \mathcal{B}} \mathrm{HH}_*(\mathcal{B}, M^{\mathrm{co}\mathcal{H}}),$$

for any centrally $\mathcal{H}$-Galois extension $\mathcal{B} \subseteq \mathcal{A}$ and any Hopf bimodule M which is symmetric as a $\mathcal{Z}$-bimodule, provided that $\mathcal{H}$ is finite-dimensional and that $\dim \mathcal{H}$ is not zero in $\mathbb{K}$. Under the same assumptions, we also show that $\mathrm{HC}_*(\mathcal{A})^{\mathrm{co}\mathcal{H}}$ and $\mathrm{HC}_*(\mathcal{B})$ are isomorphic, cf. Theorem 3.13. We conclude the paper by indicating a method to produce examples of centrally Hopf-Galois extensions of non-commutative algebras.

1. Preliminaries

In order to state and prove our main result we need several basic facts concerning Hochschild homology of Hopf-Galois extensions. Those that are well-known will be only stated, for details the reader being referred to [JŞ, SS, §1, §2].

1.1. Let $\mathcal{H}$ be a Hopf algebra with comultiplication $\Delta_\mathcal{H}$ and counit $\varepsilon_\mathcal{H}$. To denote the element $\Delta_\mathcal{H}(h)$ we shall use the Σ-notation

$$\Delta_\mathcal{H}(h) = \sum h_{(1)} \otimes h_{(2)}.$$

Similarly, for a left $\mathcal{H}$-comodule (N, ρ_N) and a right $\mathcal{H}$-comodule (M, ρ_M) we shall write

$$\rho_N(n) := \sum n_{\langle -1 \rangle} \otimes n_{\langle 0 \rangle} \qquad \text{and} \qquad \rho_M(m) = \sum m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle}.$$

For (M, ρ_M) as above we define the set of coinvariant elements in M by

$$M^{\mathrm{co}\mathcal{H}} := \{m \in M \mid \rho_M(m) = m \otimes 1\}.$$

Recall that a comodule algebra is an algebra $\mathcal{A}$ which is a right $\mathcal{H}$-comodule via a morphism of *algebras* $\rho_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{H}$. Equivalently, $(\mathcal{A}, \rho_{\mathcal{A}})$ is an $\mathcal{H}$-comodule algebra if and only if $\rho_{\mathcal{A}}(1) = 1 \otimes 1$ and

$$\rho_A(ab) = \sum a_{\langle 0 \rangle} b_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle} b_{\langle 1 \rangle},$$

for any a, b in $\mathcal{A}$. The set $\mathcal{A}^{\mathrm{co}\mathcal{H}}$ is a subalgebra in $\mathcal{A}$. If there is no danger of confusion we shall also denote this subalgebra by $\mathcal{B}$ and we shall say that $\mathcal{B} \subseteq \mathcal{A}$ is an $\mathcal{H}$-comodule algebra.

For an $\mathcal{H}$-comodule algebra $\mathcal{A}$, the category $\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$ of right Hopf modules is defined as follows. An object in $\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$ is a right $\mathcal{A}$-module M together with a right $\mathcal{H}$-coaction $\rho_M : M \rightarrow M \otimes \mathcal{H}$ such that, for any $m \in M$ and $a \in \mathcal{A}$, the following compatibility relation is verified

$$\rho_M(ma) = \sum m_{\langle 0 \rangle} a_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle} a_{\langle 1 \rangle}. \tag{7}$$

Obviously, a morphism in $\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$ is a map which is both $\mathcal{A}$-linear and $\mathcal{H}$-colinear.

The category $_{\mathcal{A}}\mathfrak{M}^{\mathcal{H}}$ is defined similarly. A left $\mathcal{A}$-module and right $\mathcal{H}$-comodule (M, ρ_M) is a left Hopf module if, for any $m \in M$ and $a \in \mathcal{A}$,

$$\rho_M(am) = \sum a_{\langle 0 \rangle} m_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle} m_{\langle 1 \rangle}. \tag{8}$$

By definition, a Hopf bimodule is an $\mathcal{A}$-bimodule M together with a right $\mathcal{H}$-coaction ρ_M such that relations (7) and (8) are satisfied for all $m \in M$ and $a \in \mathcal{A}$. A morphism between two Hopf bimodules is, by definition, a map of $\mathcal{A}$-bimodules and $\mathcal{H}$-comodules. The category of Hopf bimodules will be denoted by $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$. For example, $\mathcal{A}$ is a Hopf bimodule.

1.2. Let $\mathcal{B} \subseteq \mathcal{A}$ be an $\mathcal{H}$-comodule algebra. Recall that $\mathcal{B} \subseteq \mathcal{A}$ is an $\mathcal{H}$-Galois extension if the canonical $\mathbb{K}$-linear map

$$\beta : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{H}, \ \beta(a \otimes x) = \sum a x_{\langle 0 \rangle} \otimes x_{\langle 1 \rangle}$$

is bijective. Note that $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$ is an object in $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$ with respect to the canonical bimodule structure and the $\mathcal{H}$-coaction defined by $\mathcal{A} \otimes_{\mathcal{B}} \rho_{\mathcal{A}}$. One can also regard $\mathcal{A} \otimes \mathcal{H}$ as an object in $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$ with the $\mathcal{A}$-bimodule structure

$$a \cdot (x \otimes h) \cdot a' = \sum a x a'_{\langle 0 \rangle} \otimes h a'_{\langle 1 \rangle},$$

and the $\mathcal{H}$-coaction defined by $\mathcal{A} \otimes \Delta_{\mathcal{H}}$. With respect to these Hopf bimodule structures β is a morphism in $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$.

DEFINITION 1.3. For a $\mathbb{K}$-algebra $\mathcal{R}$ and an $\mathcal{R}$-bimodule X we define

$$X_{\mathcal{R}} := X/[\mathcal{R}, X] \qquad \text{and} \qquad X^{\mathcal{R}} := \{x \in X \mid rx = xr, \forall r \in \mathcal{R}\},$$

where $[\mathcal{R}, X]$ is the $\mathbb{K}$-subspace of X generated by all commutators $rx - xr$, with $r \in \mathcal{R}$ and $x \in X$. The class of $x \in X$ in $X_{\mathcal{R}}$ will be denoted by $[x]_{\mathcal{R}}$.

REMARK 1.4. If X is an $\mathcal{R}$-bimodule then $X_{\mathcal{R}} \cong \mathcal{R} \otimes_{\mathcal{R}^e} X$, where $\mathcal{R}^e := \mathcal{R} \otimes \mathcal{R}^{op}$ denotes the enveloping algebra of $\mathcal{R}$. The isomorphism is given by $[x]_{\mathcal{R}} \mapsto 1 \otimes_{\mathcal{R}^e} x$.

1.5. Let now $\mathcal{B} \subseteq \mathcal{A}$ be an arbitrary extension of algebras. By [JȘ, p. 145] it follows that $(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A})^{\mathcal{B}}$ is an associative algebra with the multiplication given by

$$zz' = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i a'_j \otimes_{\mathcal{B}} b'_j b_i, \tag{9}$$

where $z = \sum_{i=1}^{n} a_i \otimes_{\mathcal{B}} b_i$ and $z' = \sum_{j=1}^{m} a'_j \otimes_{\mathcal{B}} b'_j$ are arbitrary elements in $(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A})^{\mathcal{B}}$. Moreover, if M is an $\mathcal{A}$-bimodule then $M_{\mathcal{B}}$ is a right $(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A})^{\mathcal{B}}$-module with respect to the action that, for m in M and $z = \sum_{i=1}^{n} a_i \otimes_{\mathcal{B}} b_i$ in $(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A})^{\mathcal{B}}$, is defined by

$$[m]_{\mathcal{B}} \cdot z = \sum_{i=1}^{n} [b_i m a_i]_{\mathcal{B}}. \tag{10}$$

1.6. Suppose now that $\mathcal{B} \subseteq \mathcal{A}$ is an $\mathcal{H}$-Galois extension and let M be an $\mathcal{A}$-bimodule. Let $i : \mathcal{H} \longrightarrow \mathcal{A} \otimes \mathcal{H}$ denote the canonical map $i(h) = 1 \otimes h$. Following [JŞ, p. 146] we define

$$\kappa : \mathcal{H} \longrightarrow (\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A})^{\mathcal{B}}, \quad \kappa := \beta^{-1} \circ i.$$

For $h \in \mathcal{H}$ we shall use the notation $\kappa(h) = \sum \kappa^1(h) \otimes_{\mathcal{B}} \kappa^2(h)$. Thus, by definition,

$$\sum \kappa^1(h) \kappa^2(h)_{\langle 0 \rangle} \otimes \kappa^2(h)_{\langle 1 \rangle} = 1 \otimes h. \tag{11}$$

By [JŞ, p. 146], κ is an anti–morphism of algebras. Hence

$$h \cdot [m]_{\mathcal{B}} = \sum [\kappa^2(h) m \kappa^1(h)]_{\mathcal{B}} \tag{12}$$

defines a left $\mathcal{H}$-action on $M_{\mathcal{B}}$. Obviously this structure is functorial in M, so we get a functor $(-)_{\mathcal{B}} : {}_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}} \to {}_{\mathcal{H}}\mathfrak{M}$.

1.7. Let $\mathcal{H}$ be a Hopf algebra with multiplication m and comultiplication Δ. Let $\tau : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ denote the usual flip map $x \otimes y \mapsto y \otimes x$. One can prove that

$$\mathcal{C}_{\mathcal{H}} := \operatorname{coker}(m - m \circ \tau)$$

is a quotient coalgebra of $\mathcal{H}$, as the linear space generated by all commutators in $\mathcal{H}$ is a coideal of $\mathcal{H}$. The canonical projection onto $\mathcal{C}_{\mathcal{H}}$ will be denoted by $\pi_{\mathcal{H}}$. Note that $\pi_{\mathcal{H}}$ is a trace map, that is $\pi_{\mathcal{H}}(hk) = \pi_{\mathcal{H}}(kh)$ for all h and k in $\mathcal{H}$.

Dually,

$$\mathcal{R}_{\mathcal{H}} := \ker(\Delta - \tau \circ \Delta) = \{r \in \mathcal{H} \mid \sum r_{(1)} \otimes r_{(2)} = \sum r_{(2)} \otimes r_{(1)}\}$$

is a subalgebra of $\mathcal{H}$. It is clear that

$$\mathcal{R}_{\mathcal{H}} = \{r \in \mathcal{H} \mid \sum r_{(1)} \otimes r_{(2)} \otimes r_{(3)} = \sum r_{(2)} \otimes r_{(3)} \otimes r_{(1)}\}. \tag{13}$$

1.8. If $\mathcal{C}$ is a coalgebra and $\mathcal{R}$ is an algebra we define the category ${}_{\mathcal{R}}\mathfrak{M}^{\mathcal{C}}$ as follows. The objects in ${}_{\mathcal{R}}\mathfrak{M}^{\mathcal{C}}$ are left $\mathcal{R}$-modules and right $\mathcal{C}$-comodules such that the map ρ_M that defines the coaction on M is $\mathcal{R}$-linear, that is

$$\rho_M(rm) = \sum r m_{\langle 0 \rangle} \otimes m_{\langle 1 \rangle}.$$

A map $f : M \to N$ is a morphism in ${}_{\mathcal{R}}\mathfrak{M}^{\mathcal{C}}$ if it is $\mathcal{R}$-linear and $\mathcal{C}$-colinear.

For a right $\mathcal{C}$-comodule (M, ρ_M) and a left $\mathcal{C}$-comodule (V, ρ_V) we define their cotensor product by

$$M \square_{\mathcal{C}} V := \ker(\rho_M \otimes N - M \otimes \rho_V),$$

Recall that V is said to be *coflat* if the functor $(-)\square_{\mathcal{C}} V : \mathcal{M}^{\mathcal{C}} \to {}_{\mathbb{K}}\mathcal{M}$ is exact. By [DNR, Theorem 2.4.17] V is coflat if and only if V is an injective object in the category of left $\mathcal{C}$-comodules.

Note that, if $M \in {}_{\mathcal{R}}\mathfrak{M}^{\mathcal{C}}$ and $V \in {}^{\mathcal{C}}\mathfrak{M}$, then $M \square_{\mathcal{C}} V$ is an $\mathcal{R}$-submodule of $M \otimes V$, as $M \square_{\mathcal{C}} V$ is the kernel of an $\mathcal{R}$-linear map. Dually, for a right $\mathcal{R}$-module X and an object M in ${}_{\mathcal{R}}\mathfrak{M}^{\mathcal{C}}$, the tensor product $X \otimes_{\mathcal{R}} M$ is a quotient $\mathcal{C}$-comodule of $X \otimes M$.

In some special cases the cotensor product and the tensor product "commute". For instance, if X is a right $\mathcal{R}$-module, V is a left $\mathcal{C}$-comodule and $M \in {}_\mathcal{R}\mathfrak{M}^\mathcal{C}$ then

$$(X \otimes_\mathcal{R} M)\square_\mathcal{C}V \cong X \otimes_\mathcal{R} (M\square_\mathcal{C}V), \qquad (14)$$

provided that either X is flat or V is injective.

1.9. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two abelian categories and assume that, for each $n \in \mathbb{N}$, a functor $T_n : \mathfrak{A} \longrightarrow \mathfrak{B}$ is given. We say that T_* is a *homological δ-functor* if, for every short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0 \qquad (15)$$

in $\mathfrak{A}$ and $n > 0$, there are "connecting" morphism $\delta_n : T_n(X'') \to T_{n-1}(X')$ such that

$$\cdots \longrightarrow T_n(X') \longrightarrow T_n(X) \longrightarrow T_n(X'') \xrightarrow{\delta_n} T_{n-1}(X') \longrightarrow \cdots$$

is exact and functorial in the sequence in (15). Furthermore, T_* is said to be *effaceable* if, for each object X in $\mathfrak{A}$, there is an object P in $\mathfrak{A}$ together with an epimorphism from P to X such that $T_n(P) = 0$ for any $n > 0$. A morphism of δ-functors T_* and S_* with connecting homomorphisms δ_* and ∂_*, respectively, is a sequence of natural transformations $\phi_* : T_* \to S_*$ such that, for $n > 0$,

$$\phi_{n-1} \circ \delta_n = \partial_n \circ \phi_n.$$

By Theorem 7.5 in [Br, Chapter III], homological and effaceable δ-functors have the following universal property. If T_* and S_* are homological and effaceable δ-functors and $\phi_0 : T_0 \to S_0$ is a natural transformation, then there is a unique morphism of δ-functors $\phi_* : T_* \to S_*$ that lifts ϕ_0.

PROPOSITION 1.10. *Let $\mathcal{H}$ be a Hopf algebra with antipode $S_\mathcal{H}$. Suppose that $\mathcal{B} \subseteq \mathcal{A}$ is an $\mathcal{H}$-Galois extension, (M, ρ_M) is a Hopf bimodule and V is a left $\mathcal{H}$-comodule.*

(1) *There is a $\mathbb{K}$-linear map $\rho_0(M) : M_\mathcal{B} \to M_\mathcal{B} \otimes \mathcal{H}$ such that $(M_\mathcal{B}, \rho_0(M))$ is a quotient $\mathcal{H}$-comodule of (M, ρ_M). Moreover, for $m \in M$ and $h \in \mathcal{H}$ we have*

$$\rho_0(M)(h \cdot [m]_\mathcal{B}) = \sum h_{(2)} \cdot [m_{\langle 0\rangle}]_\mathcal{B} \otimes h_{(3)}m_{\langle 1\rangle}S_\mathcal{H}h_{(1)}. \qquad (16)$$

(2) *If $S_\mathcal{H}$ is an involution, then $(M_\mathcal{B} \otimes \pi_\mathcal{H}) \circ \rho_0(M) : M_\mathcal{B} \to M_\mathcal{B} \otimes \mathcal{C}_\mathcal{H}$ is a morphism of left $\mathcal{R}_\mathcal{H}$-modules. Hence, with respect to the above $\mathcal{C}_\mathcal{H}$-comodule structure, $M_\mathcal{B}$ is an object in ${}_{\mathcal{R}_\mathcal{H}}\mathfrak{M}^{\mathcal{C}_\mathcal{H}}$ and, for any right $\mathcal{R}_\mathcal{H}$-module X, the coalgebra $\mathcal{C}_\mathcal{H}$ coacts canonically on $X \otimes_{\mathcal{R}_\mathcal{H}} M_\mathcal{B}$. The $\mathcal{C}_\mathcal{H}$-coaction on $M_\mathcal{B}$ will be denoted by $\rho_0(M)$ too.*

(3) *If V is a left $\mathcal{C}_\mathcal{H}$-comodule then $M\square_{\mathcal{C}_\mathcal{H}}V$ is a $\mathcal{B}^e$-submodule of $M \otimes V$. Under the additional assumption that V is injective, $(M\square_{\mathcal{C}_\mathcal{H}}V)_\mathcal{B}$ and $M_\mathcal{B}\square_{\mathcal{C}_\mathcal{H}}V$ are isomorphic linear spaces. In particular, the action of $\mathcal{R}_\mathcal{H}$ on the latter vector space can be transported to $(M\square_{\mathcal{C}_\mathcal{H}}V)_\mathcal{B}$.*

Proof. (1) Obviously, $(M_\mathcal{B}, \rho_0(M))$ is a quotient $\mathcal{H}$-comodule of M, as $[\mathcal{B}, M]$ is a subcomodule of M. For $M = \mathcal{A}$, identity (16) is proven in [JŞ, Proposition 2.6]. The general case can be handled in a similar manner, replacing $a \in \mathcal{A}$ by $m \in M$ everywhere in the proof of [JŞ, Relation (6)].

(2) Recall that $S_\mathcal{H}$ is an involution, i.e. $S_\mathcal{H}^2 = \mathrm{Id}_\mathcal{H}$, if and only if

$$\sum r_{(2)}S_\mathcal{H}r_{(1)} = \sum S_\mathcal{H}r_{(2)}r_{(1)} = \varepsilon(r)1_\mathcal{H}. \qquad (17)$$

Clearly $(M_\mathcal{B} \otimes \pi_\mathcal{H}) \circ \rho_0(M)$ defines a $\mathcal{C}_\mathcal{H}$-comodule structure on $M_\mathcal{B}$. For brevity we shall denote this map by $\rho_0(M)$ too. For $r \in \mathcal{R}_\mathcal{H}$ and $m \in M$ we get

$$
\begin{aligned}
\rho_0(M)(r \cdot [m]_0(M)) &= \sum r_{(2)} \cdot [m_{\langle 0 \rangle}]_\mathcal{B} \otimes \pi_\mathcal{H}(r_{(3)} m_{\langle 1 \rangle} S_\mathcal{H} r_{(1)}) \\
&= \sum r_{(2)} \cdot [m_{\langle 0 \rangle}]_\mathcal{B} \otimes \pi_\mathcal{H}(m_{\langle 1 \rangle} S_\mathcal{H} r_{(1)} r_{(3)}) \\
&= \sum r_{(3)} \cdot [m_{\langle 0 \rangle}]_\mathcal{B} \otimes \pi_\mathcal{H}(m_{\langle 1 \rangle} r_{(2)} S_\mathcal{H} r_{(1)}) \\
&= \sum r \cdot [m_{\langle 0 \rangle}]_\mathcal{B} \otimes \pi_\mathcal{H}(m_{\langle 1 \rangle}) \\
&= r \cdot \rho_0(M)([m]_\mathcal{B}).
\end{aligned}
$$

Note that the second and the third equalities are consequences of the fact that $\pi_\mathcal{H}$ is a trace map and respectively of relation (13). To deduce the penultimate identity we use (17). In conclusion, $\rho_0(M)$ is a morphism of $\mathcal{R}_\mathcal{H}$-modules. Hence M is an object in ${}_{\mathcal{R}_\mathcal{H}}\mathfrak{M}^{\mathcal{C}_\mathcal{H}}$ and, in view of §1.8, one can regard $M_\mathcal{B} \otimes_\mathcal{R} X$ as a quotient comodule of $M_\mathcal{B} \otimes X$.

(3) Obviously, $\rho' := (M \otimes \pi_\mathcal{H}) \circ \rho$ defines a $\mathcal{C}_\mathcal{H}$-coaction on M and it is a morphism of $\mathcal{B}$-bimodules, as ρ is so. Thus (M, ρ') is an object in ${}_{\mathcal{B}^e}\mathfrak{M}^{\mathcal{C}_\mathcal{H}}$ and $M \square_{\mathcal{C}_\mathcal{H}} V$ is a $\mathcal{B}^e$-submodule of $M \otimes V$, cf. §1.8. If V is an injective $\mathcal{C}_\mathcal{H}$-comodule, then

$$
(M \square_{\mathcal{C}_\mathcal{H}} V)_\mathcal{B} \cong B \otimes_{\mathcal{B}^e} (M \square_{\mathcal{C}_\mathcal{H}} V) \cong (B \otimes_{\mathcal{B}^e} M) \square_{\mathcal{C}_\mathcal{H}} V \cong M_\mathcal{B} \square_{\mathcal{C}_\mathcal{H}} V. \tag{18}
$$

Since $M_\mathcal{B}$ is an object in ${}_{\mathcal{R}_\mathcal{H}}\mathfrak{M}^{\mathcal{C}_\mathcal{H}}$, it follows that $M_\mathcal{B} \square_{\mathcal{C}_\mathcal{H}} V$ is an $\mathcal{R}_\mathcal{H}$-submodule of $M_\mathcal{B} \otimes V$. In particular, $M_\mathcal{B} \square_{\mathcal{C}_\mathcal{H}} V$ is an $\mathcal{R}_\mathcal{H}$-module. To conclude the proof, we take on $(M \square_{\mathcal{C}_\mathcal{H}} V)_\mathcal{B}$ the unique $\mathcal{R}_\mathcal{H}$-action that makes the composition of the isomorphisms in (18) an $\mathcal{R}_\mathcal{H}$-linear map. $\qquad\square$

REMARK 1.11. We keep the assumptions in the third part of Proposition 1.10. Let $z := \sum_{i=1}^n m_i \otimes v_i$ be an element in $M \square_{\mathcal{C}_\mathcal{H}} V$. The composition of the $\mathcal{R}_\mathcal{H}$-linear isomorphisms in (18) maps $[z]_\mathcal{B}$ to $\sum_{i=1}^n [m_i]_\mathcal{B} \otimes v_i$. Obviously, this isomorphism is natural in $M \in {}_\mathcal{A}\mathfrak{M}_\mathcal{A}^\mathcal{H}$. It is not hard to see that, for $h \in \mathcal{R}_\mathcal{H}$,

$$
h \cdot [z]_\mathcal{B} = \sum_{i=1}^n [\kappa^2(h) m_i \kappa^1(h) \otimes v_i]_\mathcal{B}.
$$

2. THE SPECTRAL SEQUENCE

In this section, given an $\mathcal{H}$-Galois extension $\mathcal{B} \subseteq \mathcal{A}$, a Hopf bimodule M and an injective left $\mathcal{C}_\mathcal{H}$-comodule V, we construct a spectral sequence that converges to $\mathrm{H}_*(\mathcal{A}, M) \square_{\mathcal{C}_\mathcal{H}} V$. Our result, Theorem 2.23, will be obtained as a direct application of a variant of Grothendieck's spectral sequence, which will be deduced from the following two lemmas and [We, Corollary 5.8.4].

Recall that a category $\mathfrak{A}$ is cocomplete if and only if any set of objects in $\mathfrak{A}$ has a direct sum. If X is an object in a category, then we shall also write X for the identity map of X.

LEMMA 2.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be cocomplete abelian categories and let H, $H' : \mathfrak{A} \to \mathfrak{B}$ be two right exact functors that commute with direct sums. If U is a generator in $\mathfrak{A}$ and there is a natural morphism $\phi : H \to H'$ such that $\phi(U)$ is an isomorphism, then $\phi(X)$ is an isomorphism, for every object X in $\mathfrak{A}$.*

Proof. Let X be an object in $\mathfrak{A}$. Since U is a generator in $\mathfrak{A}$, there is an exact sequence

$$U^{(J)} \xrightarrow{u} U^{(I)} \xrightarrow{v} X \longrightarrow 0,$$

where I and J are certain sets. Hence in the following diagram

$$
\begin{array}{ccccccc}
H(U^{(J)}) & \xrightarrow{H(u)} & H(U^{(I)}) & \xrightarrow{H(v)} & H(X) & \longrightarrow & 0 \\
{\scriptstyle\phi(U^{(J)})}\downarrow & & {\scriptstyle\phi(U^{(I)})}\downarrow & & \downarrow{\scriptstyle\phi(X)} & & \\
H'(U^{(J)}) & \xrightarrow[H'(u)]{} & H'(U^{(I)}) & \xrightarrow[H'(v)]{} & H'(X) & \longrightarrow & 0
\end{array}
$$

the squares are commutative and the lines are exact. Recall that H commutes with direct sums if the canonical map $\alpha : \oplus_{i\in I} H(X_i) \to H(\oplus_{i\in I} X_i)$ is an isomorphism for each family of objects $(X_i)_{i\in I}$ in $\mathfrak{A}$. Now one can see easily that $\phi(U^{(I)})$ and $\phi(U^{(J)})$ are isomorphisms, as H and H' commute with direct sums and $\phi(U)$ is an isomorphism. Thus $\phi(X)$ is an isomorphism too. $\qquad\square$

LEMMA 2.2. *Let $\mathfrak{A}$, $\mathfrak{B}$ and $\mathcal{C}$ be cocomplete abelian categories with enough projective objects. Let $F : \mathcal{B} \to \mathfrak{C}$ and $G : \mathfrak{A} \to \mathfrak{B}$ be right exact functors that commute with direct sums. If U is a generator in $\mathfrak{A}$ such that $G(U)$ is F-acyclic, then $G(P)$ is F-acyclic for any projective object P in $\mathfrak{A}$.*

Proof. Recall that $G(U)$ is F-acyclic if $\mathrm{L}_n F(G(U)) = 0$ for any $n > 0$. Let P be a projective object in $\mathfrak{A}$. There is a set I such that P is a direct summand of $U^{(I)}$. Hence $G(P)$ is a direct summand of $G(U^{(I)})$. On the other hand, the proof of [We, Corollary 2.6.11] works for any functor that commutes with direct sums. Thus $\mathrm{L}_n F$ commutes with direct sums, so $G(U^{(I)}) \cong G(U)^{(I)}$ is F-acyclic. Then $G(P)$ is also F-acyclic. $\qquad\square$

PROPOSITION 2.3. *Let $G : \mathfrak{A} \to \mathfrak{B}$, $F : \mathfrak{B} \to \mathfrak{C}$ and $H : \mathfrak{A} \to \mathfrak{C}$ be right exact functors that commute with direct sums, where $\mathfrak{A}$, $\mathfrak{B}$ and $\mathcal{C}$ are cocomplete abelian categories with enough projective objects. Assume that U is a generator in $\mathfrak{A}$ and that $\phi : F \circ G \to H$ is a natural transformation. If $\phi(U)$ is an isomorphism and $G(U)$ is F-acyclic then, for every object X in $\mathfrak{A}$, there is a functorial spectral sequence*

$$\mathrm{L}_p F(\mathrm{L}_q G(X)) \Longrightarrow \mathrm{L}_{p+q} H(X). \tag{19}$$

Proof. By Lemma 2.1, for every object X the morphism $\phi(X)$ is an isomorphism. On the other hand, if X is projective in $\mathfrak{A}$ then $\mathrm{L}_n F(G(X)) = 0$ for every $n \in \mathbb{N}^*$. Hence, we obtain (19) as a particular case of [We, Corollary 5.8.4]. $\qquad\square$

We take $\mathcal{B} \subseteq \mathcal{A}$ to be a faithfully flat Galois extension. For proving Theorem 2.23, one of our main results, we shall apply Proposition 2.3. In order to do that we need some properties of the category ${}_{\mathcal{A}}\mathcal{M}^{\mathcal{H}}_{\mathcal{A}}$. We start with the following.

PROPOSITION 2.4. *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode. Let $\mathcal{B} \subseteq \mathcal{A}$ be a faithfully flat $\mathcal{H}$-Galois extension. Then $\mathcal{A} \otimes \mathcal{A}$ is a projective generator in the category of Hopf bimodules. It is also projective as a $\mathcal{B}$-bimodule.*

Proof. By [SS, Theorem 4.10] the induction functor $(-) \otimes_{\mathcal{B}} \mathcal{A} : \mathfrak{M}_{\mathcal{B}} \to \mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$ is an equivalence of categories and its inverse is $(-)^{\text{co}\mathcal{H}}$. We deduce that, for an arbitrary right Hopf module X, the canonical map $X^{\text{co}\mathcal{H}} \otimes_{\mathcal{B}} \mathcal{A} \to X$ induced by the module structure of X is an isomorphism of right Hopf modules. Let M be a Hopf bimodule. Hence, the $\mathcal{A}$-bimodule structure on M defines an epimorphism $\mathcal{A} \otimes M^{\text{co}\mathcal{H}} \otimes \mathcal{A} \to M$ of Hopf bimodules. Thus $\mathcal{A} \otimes \mathcal{A}$ is a generator in the category $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$.

Let $p : X \to Y$ be an epimorphism of Hopf bimodules and $f : \mathcal{A} \otimes \mathcal{A} \to Y$ be an arbitrary morphism in $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$. We want to show that there is a morphism $g : \mathcal{A} \otimes \mathcal{A} \to X$ of Hopf bimodules such that $p \circ g = f$. Indeed, if $y := f(1_{\mathcal{A}} \otimes 1_{\mathcal{A}})$ then $y \in Y^{\text{co}\mathcal{H}}$. Since $(-)^{\text{co}\mathcal{H}} : \mathfrak{M}_{\mathcal{A}}^{\mathcal{H}} \to \mathfrak{M}_{\mathcal{B}}$ is an equivalence of categories it follows that $(-)^{\text{co}\mathcal{H}}$ is exact. Hence $p(X^{\text{co}\mathcal{H}}) = Y^{\text{co}\mathcal{H}}$. Let $x \in X^{\text{co}\mathcal{H}}$ be an element such that $p(x) = y$. There is a unique morphism of $\mathcal{A}$-bimodules $g : \mathcal{A} \otimes \mathcal{A} \to X$ such that $g(a' \otimes a'') = a'xa''$. Since x is an $\mathcal{H}$-coinvariant element in X, one can check easily that g is a map of $\mathcal{H}$-comodules too. Obviously, $p \circ g = f$.

By [SS, Theorems 4.9 and 4.10] $\mathcal{A}$ is projective as a left and right $\mathcal{B}$-module. Thus $\mathcal{A}^e$ is projective as a left $\mathcal{B}^e$-module, that is $\mathcal{A} \otimes \mathcal{A}$ is a projective $\mathcal{B}$-bimodule. $\qquad\square$

COROLLARY 2.5. *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode. If $\mathcal{B} \subseteq \mathcal{A}$ is a faithfully flat $\mathcal{H}$-Galois extension then $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$ has enough projective objects.*

Proof. Every category with a projective generator has enough projective objects. $\qquad\square$

DEFINITION 2.6. For a $\mathbb{K}$-algebra $\mathcal{R}$ and an $\mathcal{R}$-bimodule X, let $(C_*(\mathcal{R}, X), b_*)$ be the chain complex given by $C_n(\mathcal{R}, X) = X \otimes \mathcal{R}^{\otimes n}$ and

$$b_n(x \otimes r_1 \otimes \cdots \otimes r_n) = xr_1 \otimes \cdots \otimes r_n + \sum_{i=1}^{n-1}(-1)^i x \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n$$
$$+ (-1)^n r_n x \otimes r_1 \otimes \cdots \otimes r_{n-1}.$$

Hochschild homology of $\mathcal{R}$ with coefficients in X is, by definition, the homology of $(C_*(\mathcal{R}, X), b_*)$. It will be denoted by $\text{HH}_*(\mathcal{R}, X)$.

2.7. Let $\mathcal{R}$ and X be as in the above definition. It is well-known that Hochschild homology of $\mathcal{R}$ with coefficients in X may be defined in an equivalent way by

$$\text{HH}_*(\mathcal{R}, X) = \text{Tor}_*^{\mathcal{R}^e}(\mathcal{R}, X).$$

Since $X_{\mathcal{R}} \cong \mathcal{R} \otimes_{\mathcal{R}^e} X$, it also follows that $\text{HH}_*(\mathcal{R}, -)$ are the left derived functors of $(-)_{\mathcal{R}} : _{\mathcal{R}}\mathfrak{M}_{\mathcal{R}} \to \mathfrak{M}_{\mathbb{K}}$.

2.8. Let $\mathcal{B} \subseteq \mathcal{A}$ be an $\mathcal{H}$-comodule algebra and M be a Hopf bimodule. Obviously, $\rho_n(M) : C_n(\mathcal{B}, M) \to C_n(\mathcal{B}, M) \otimes \mathcal{H}$ given by

$$\rho_n(M)(m \otimes b_1 \otimes \cdots \otimes b_n) = \sum (m_{\langle 0 \rangle} \otimes b_1 \otimes \cdots \otimes b_n) \otimes m_{\langle 1 \rangle}.$$

defines a comodule structure on $C_n(\mathcal{B}, M)$ such that $C_*(\mathcal{B}, M)$ is a complex of right $\mathcal{H}$-comodules. Note that, if $\mathcal{Z}$ is the center of $\mathcal{A}$ then $C_*(\mathcal{B}, M)$ is a complex of left $\mathcal{Z}_0$-modules, where $\mathcal{Z}_0 := \mathcal{Z} \bigcap \mathcal{B}$. Indeed, $\mathcal{Z}_0$-acts on $M \otimes \mathcal{B}^{\otimes n}$ by

$$z \cdot (m \otimes b_1 \otimes \cdots \otimes b_n) = (z \cdot m) \otimes b_1 \otimes \cdots \otimes b_n,$$

and the differential maps b_* are morphisms of $\mathcal{Z}_0$-modules. Clearly, $\rho_n(M)$ is a morphism of $\mathcal{Z}_0$-modules, so $C_n(\mathcal{B}, -)$ can be seen as a functor from $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$ to the category of chain complexes in $_{\mathcal{Z}_0}\mathfrak{M}^{\mathcal{H}}$. Therefore, a fortiori, the functors $\text{HH}_*(\mathcal{B}, -)$ map a

Hopf bimodule to an object in $_{\mathcal{Z}_0}\mathfrak{M}^{\mathcal{H}}$. The $\mathcal{H}$-coaction on $\mathrm{HH}_*(\mathcal{B}, M)$ will still be denoted by $\rho_*(M)$.

REMARK 2.9. By definition, Hochschild homology of $\mathcal{B}$ with coefficients in M in degree zero equals $M_{\mathcal{B}}$. Thus in Proposition 1.10 (1) and §2.8, we constructed two $\mathcal{H}$-coactions on $M_{\mathcal{B}}$, both of them being denoted by $\rho_0(M)$. The notation we have used is consistent, as these coactions are identical.

LEMMA 2.10. *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode. If $\mathcal{B} \subseteq \mathcal{A}$ is a faithfully flat $\mathcal{H}$-Galois extension then $\mathrm{HH}_*(\mathcal{B}, -) : {}_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}} \to {}_{\mathcal{Z}_0}\mathfrak{M}^{\mathcal{H}}$ is a homological and effaceable δ-functor, where $\mathcal{Z}$ is the center of $\mathcal{A}$ and $\mathcal{Z}_0 := \mathcal{Z} \bigcap \mathcal{B}$.*

Proof. We take a short exact sequence of Hopf bimodules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0. \tag{20}$$

We have to prove that there are connecting maps $\delta_n : \mathrm{HH}_n(\mathcal{B}, M'') \to \mathrm{HH}_{n-1}(\mathcal{B}, M')$, which are homomorphisms of $\mathcal{Z}_0$-modules and $\mathcal{H}$-comodules, making

$$\cdots \longrightarrow \mathrm{HH}_n(\mathcal{B}, M') \longrightarrow \mathrm{HH}_n(\mathcal{B}, M) \longrightarrow \mathrm{HH}_n(\mathcal{B}, M'') \xrightarrow{\delta_n} \mathrm{HH}_{n-1}(\mathcal{B}, M') \longrightarrow \cdots$$

a functorial exact sequence. In our setting, the connecting maps are obtained by applying the long exact sequence in homology to the following short exact sequence of complexes in $_{\mathcal{Z}_0}\mathfrak{M}^{\mathcal{H}}$

$$0 \longrightarrow C_*(\mathcal{B}, M') \longrightarrow C_*(\mathcal{B}, M) \longrightarrow C_*(\mathcal{B}, M'') \longrightarrow 0.$$

Let us prove that $\mathrm{HH}_*(\mathcal{B}, -)$ is effaceable too. Let M be a given Hopf bimodule. By Proposition 2.4, there exists a certain set I such that M is the quotient of $P := (\mathcal{A} \otimes \mathcal{A})^{(I)}$ as a Hopf bimodule. In view of the same proposition, $\mathcal{A} \otimes \mathcal{A}$ is projective as a $\mathcal{B}$-bimodule. Thus, for $n > 0$,

$$\mathrm{HH}_n(\mathcal{B}, P) \cong \mathrm{Tor}_n^{\mathcal{B}^e}(\mathcal{B}, P) = 0.$$

Hence the lemma is completely proven. $\qquad\square$

REMARK 2.11. Both $\mathrm{HH}_*(\mathcal{B}, -)$ and $\mathrm{HH}_*(\mathcal{B}, -) \otimes \mathcal{H}$ can be seen as homological and effaceable δ-functors that map a Hopf bimodule to an object in $_{\mathcal{Z}_0}\mathfrak{M}^{\mathcal{H}}$. The natural transformations $\rho_*(-)$ in §2.8 define a morphisms of δ-functors that lifts

$$\rho_0(-) : \mathrm{HH}_0(\mathcal{B}, -) \longrightarrow \mathrm{HH}_0(\mathcal{B}, -) \otimes \mathcal{H}.$$

PROPOSITION 2.12. *Let $\mathcal{H}$ be a Hopf algebra with bijective antipode. We assume that $\mathcal{B} \subseteq \mathcal{A}$ is a faithfully flat $\mathcal{H}$-Galois extension and that M is a Hopf bimodule.*

   (1) *There is an $\mathcal{H}$-action on $\mathrm{HH}_n(\mathcal{B}, M)$ that extends the module structure defined in (12). Moreover, for any $h \in \mathcal{H}$ and $\omega \in \mathrm{HH}_n(\mathcal{B}, M)$,*

$$\rho_n(M)(h \cdot \omega) = \sum h_{(2)} \cdot \omega_{\langle 0 \rangle} \otimes h_{(3)} \omega_{\langle 1 \rangle} S_{\mathcal{H}} h_{(1)}. \tag{21}$$

   (2) *If the antipode of $\mathcal{H}$ is involutive then $\mathrm{HH}_*(\mathcal{B}, -)$ is a homological and effaceable δ-functor that takes values in $_{\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0}\mathfrak{M}^{\mathcal{C}_{\mathcal{H}}}$.*

Proof. (1) We fix $h \in \mathcal{H}$. The module structure constructed in formula (12) defines a natural map

$$\mu_0^h(M) : \mathrm{HH}_0(\mathcal{B}, M) \to \mathrm{HH}_0(\mathcal{B}, M), \quad \mu_0^h(M)([m]_{\mathcal{B}}) = h \cdot [m]_{\mathcal{B}}.$$

In view of Lemma 2.10 the δ-functor $\mathrm{HH}_*(\mathcal{B}, -) : {}_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}} \to {}_{\mathcal{Z}_0}\mathfrak{M}^{\mathcal{H}}$ is homological and effaceable. Hence, by the universal property of these functors (see §1.9) there is a unique morphism of δ-functors

$$\mu_*^h(-) : \mathrm{HH}_*(\mathcal{B}, -) \to \mathrm{HH}_*(\mathcal{B}, -)$$

that lifts $\mu_0^h(-)$. Note that, by definition, $\mu_*^h(-)$ and the connecting morphisms δ_* are morphisms of Z_0-modules and $\mathcal{H}$-comodules. For $\omega \in \mathrm{HH}_n(\mathcal{B}, M)$, we set

$$h \cdot \omega := \mu_n^h(M)(\omega).$$

Proceeding as in the proof of [S1, Proposition 2.4], one can easily see that the above formula defines a natural action of $\mathcal{H}$ on $\mathrm{HH}_n(\mathcal{B}, M)$. By construction, it lifts the action in (12). Note that, for any n, the connecting maps δ_n are morphisms of $\mathcal{H}$-modules, since $\mu_*^h(-)$ is a morphism of δ-functors.

To conclude the proof of this part, it remains to prove relation (21). We proceed by induction. In degree zero the required identity holds by (16). Let us assume that (21) holds in degree n for any Hopf bimodule. Let M be a given Hopf bimodule. We take an exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

of Hopf bimodules such that $P := (\mathcal{A} \otimes \mathcal{A})^{(I)}$. Since δ_{n+1} is a homomorphism of $\mathcal{H}$-modules and $\mathcal{H}$-comodules and using the induction hypothesis, for $h \in \mathcal{H}$ and $\omega \in \mathrm{HH}_{n+1}(\mathcal{B}, M)$, we get

$$
\begin{aligned}
(\delta_{n+1} \otimes \mathcal{H}) \left(\rho_{n+1}(M)(h \cdot \omega) \right) &= \rho_n(K) \left(\delta_{n+1}(h \cdot \omega) \right) \\
&= \rho_n(K) \left(h \cdot \delta_{n+1}(\omega) \right) \\
&= \sum h_{(2)} \cdot \delta_{n+1}(\omega_{\langle 0 \rangle}) \otimes h_{(3)} \omega_{\langle 1 \rangle} S_{\mathcal{H}} h_{(1)} \\
&= (\delta_{n+1} \otimes \mathcal{H}) \left(\sum h_{(2)} \cdot \omega_{\langle 0 \rangle} \otimes h_{(3)} \omega_{\langle 1 \rangle} S_{\mathcal{H}} h_{(1)} \right).
\end{aligned}
$$

As $\mathrm{HH}_{n+1}(\mathcal{B}, P) = 0$ it follows that δ_{n+1} is injective. Consequently, $\delta_{n+1} \otimes \mathcal{H}$ is also injective. Thus the foregoing computation implies relation (21).

(2) Let M be a given Hopf bimodule. For $z \in \mathcal{Z}_0$ we define

$$\nu_n^z(M) : \mathrm{HH}_n(\mathcal{B}, M) \to \mathrm{HH}_n(\mathcal{B}, M), \quad \nu_n^z(M)(\omega) = z \cdot \omega.$$

We claim that $\mu_*^h(-)$ and $\nu_*^z(-)$ commute for all $h \in \mathcal{H}$, i.e.

$$\mu_*^h(-) \circ \nu_*^z(-) = \nu_*^z(-) \circ \mu_*^h(-). \tag{22}$$

In degree zero this identity follows from the computation below, where for brevity we write μ_0^h and ν_0^z instead of $\mu_0^h(M)$ and $\nu_0^z(M)$. Indeed,

$$\left(\mu_0^h \circ \nu_0^z \right) ([m]_{\mathcal{B}}) = \sum [\kappa^2(h)zm\kappa^1(h)]_{\mathcal{B}} = z \cdot \sum [\kappa^2(h)m\kappa^1(h)]_{\mathcal{B}} = \left(\nu_0^z \circ \mu_0^h \right) ([m]_{\mathcal{B}}),$$

where for the second equality we used that z is a central element. Furthermore, the natural transformations that appear in the left and right hand sides of (22) are morphisms of δ-functors that lift respectively $\mu_0^h(-) \circ \nu_0^z(-)$ and $\nu_0^z(-) \circ \mu_0^h(-)$. Hence (22) follows by the universal property of homological and δ-functors. In view of the relation (22) it follows that $\mathrm{HH}_n(\mathcal{B}, M)$ is an $\mathcal{H} \otimes \mathcal{Z}_0$-module with respect to

$$(h \otimes z) \cdot \omega = \left[\mu_n^h(M) \circ \nu_n^z(M) \right] (\omega).$$

We can now prove that $\mathrm{HH}_n(\mathcal{B}, M)$ is an object in $_{\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0}\mathfrak{M}^{\mathcal{C}_{\mathcal{H}}}$. As $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$ is subalgebra of $\mathcal{H} \otimes \mathcal{Z}_0$, it acts on $\mathrm{HH}_n(\mathcal{B}, M)$. The coalgebra $\mathcal{C}_{\mathcal{H}}$ coacts on the Hochschild homology of $\mathcal{B}$ with coefficients in M via

$$\bar{\rho}_*(M) := \left(\mathrm{HH}_*(\mathcal{B}, M) \otimes \pi_{\mathcal{H}}\right) \otimes \rho(M).$$

To simplify the notation, we shall write $\rho_*(M)$ instead of $\bar{\rho}_*(M)$. We have to show that $\rho_n(M)$ is a morphism of $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-modules. By Lemma 2.10 we already know that $\rho_n(M)$ is a morphism of $\mathcal{Z}_0$-modules. Thus, it remains to check that $\rho_n(M)$ is a morphism of $\mathcal{R}_{\mathcal{H}}$-modules too. For the case $n = 0$ see the proof of Proposition 1.10 (2). In fact, the same proof works for an arbitrary n, just replacing $[m]_{\mathcal{B}}$ by an element $\omega \in \mathrm{HH}_n(\mathcal{B}, M)$ and using (21) instead of (16).

We still have to prove that $\mathrm{HH}_*(\mathcal{B}, -) : _{\mathcal{A}}\mathfrak{M}^{\mathcal{H}}_{\mathcal{A}} \to {}_{\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0}\mathfrak{M}^{\mathcal{C}_{\mathcal{H}}}$ is a homological and effaceable δ-functor, i.e. for every short exact of Hopf bimodules the corresponding connecting maps δ_* are morphisms of $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-modules and $\mathcal{H}$-comodules. By Lemma 2.10, it follows that δ_* are morphisms of $\mathcal{Z}_0$-modules and $\mathcal{H}$-comodules. By the proof of the first part of the proposition, δ_* are also morphisms of $\mathcal{H}$-modules. Hence, a fortiori, they are morphisms of $\mathcal{R}_{\mathcal{H}}$-modules. $\qquad\square$

2.13. The natural transformations that define the $\mathcal{R}_{\mathcal{H}}$-module and the $\mathcal{C}_{\mathcal{H}}$-comodule structures of $\mathrm{HH}_*(\mathcal{B}, -)$, as in the above proposition, will be denoted by $\mu_*(-)$ and $\rho_*(-)$, respectively.

Let us take an injective left $\mathcal{C}_{\mathcal{H}}$-comodule V. By §1.8, for a Hopf bimodule M, the cotensor product $\mathrm{HH}_*(\mathcal{B}, M)\square_{\mathcal{C}_{\mathcal{H}}}V$ is a left $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-module. It follows that $\mathrm{HH}_*(\mathcal{B}, -)\square_{\mathcal{C}_{\mathcal{H}}}V$ is a homological and effaceable functor from the category of Hopf bimodules to the category of left $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-modules. Of course, its connecting maps are $\delta_*\square_{\mathcal{C}_{\mathcal{H}}}V$, where δ_* are the connecting homomorphisms of the δ-functor $\mathrm{HH}_*(\mathcal{B}, -)$.

To simplify the notation, we shall denote $\mathrm{HH}_0(\mathcal{B}, -)\square_{\mathcal{C}_{\mathcal{H}}}V$ by G_V. By the foregoing observations G_V maps a Hopf bimodule to a left $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-module. Our aim now is to describe the left derived functors of G_V.

PROPOSITION 2.14. *Let $\mathcal{H}$ be a Hopf algebra such that $S^2_{\mathcal{H}} = \mathrm{Id}_{\mathcal{H}}$. If $\mathcal{B} \subseteq \mathcal{A}$ is a faithfully flat $\mathcal{H}$-Galois extension and V is an injective $\mathcal{C}_{\mathcal{H}}$-comodule, then*

$$\mathrm{HH}_*(\mathcal{B}, -\square_{\mathcal{C}_{\mathcal{H}}}V) : {}_{\mathcal{A}}\mathfrak{M}^{\mathcal{H}}_{\mathcal{A}} \longrightarrow {}_{\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0}\mathfrak{M}$$

is a homological and effaceable δ-functor. As δ-functors from $_{\mathcal{A}}\mathfrak{M}^{\mathcal{H}}_{\mathcal{A}}$ to $_{\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0}\mathfrak{M}$,

$$\mathrm{L}_*G_V \cong \mathrm{HH}_*(\mathcal{B}, -)\square_{\mathcal{C}_{\mathcal{H}}}V \cong \mathrm{HH}_*\left(\mathcal{B}, -\square_{\mathcal{C}_{\mathcal{H}}}V\right). \qquad (23)$$

Proof. By Proposition 1.10 (3) the cotensor product $M\square_{\mathcal{C}_{\mathcal{H}}}V$ is a $\mathcal{B}$-bimodule, for every Hopf bimodule M. Hence Hochschild homology of $\mathcal{B}$ with coefficients in $M\square_{\mathcal{C}_{\mathcal{H}}}V$ makes sense. We set $T_* := \mathrm{HH}_*(\mathcal{B}, -\square_{\mathcal{C}_{\mathcal{H}}}V)$ and take a short exact sequence of Hopf bimodules as in (20). Since V is an injective comodule,

$$0 \longrightarrow C_*(\mathcal{B}, M'\square_{\mathcal{C}_{\mathcal{H}}}V) \longrightarrow C_*(\mathcal{B}, M\square_{\mathcal{C}_{\mathcal{H}}}V) \longrightarrow C_*(\mathcal{B}, M''\square_{\mathcal{C}_{\mathcal{H}}}V) \longrightarrow 0$$

is exact. By the definition of T_* and the long exact sequence in homology we deduce that T_* is homological, regarded as a δ-functor to the category of vector spaces. Recall that $U := \mathcal{A} \otimes \mathcal{A}$ is a generator in the category of Hopf bimodules. Therefore, to prove that T_* is effaceable, it is enough to show that $T_n(X) = 0$, where X is an arbitrary direct sum of copies of U and $n > 0$. In fact, as Hochschild homology and

the cotensor product commute with direct sums, we may assume that $X = U$. We claim that $U\square_{\mathcal{C}_{\mathcal{H}}}V$ is flat as a $\mathcal{B}$-bimodule. By (14), for an arbitrary $\mathcal{B}$-bimodule N,

$$N \otimes_{\mathcal{B}^e} (U\square_{\mathcal{C}_{\mathcal{H}}}V) \cong (N \otimes_{\mathcal{B}^e} \mathcal{A}^e)\square_{\mathcal{C}_{\mathcal{H}}}V \cong (\mathcal{A} \otimes_{\mathcal{B}} N \otimes_{\mathcal{B}} \mathcal{A})\square_{\mathcal{C}_{\mathcal{H}}}V.$$

Hence, the functors $(-)\otimes_{\mathcal{B}^e}(U\square_{\mathcal{C}_{\mathcal{H}}}V)$ and $(-\square_{\mathcal{C}_{\mathcal{H}}}V)\circ(\mathcal{A}\otimes_{\mathcal{B}} - \otimes_{\mathcal{B}} \mathcal{A})$ are isomorphic. Since the antipode of $\mathcal{H}$ is bijective, by [SS, Theorems 4.9 and 4.10], $\mathcal{A}$ is faithfully flat as a left and a right $\mathcal{B}$-module. Therefore, $\mathcal{A}\otimes_{\mathcal{B}} - \otimes_{\mathcal{B}} \mathcal{A}$ is an exact functor. As V is injective, the functor $(-)\square_{\mathcal{C}_{\mathcal{H}}}V$ is also exact, so $U\square_{\mathcal{C}_{\mathcal{H}}}V$ is flat. Thus

$$T_n(U) \cong \mathrm{Tor}_n^{\mathcal{B}^e}(\mathcal{B}, U\square_{\mathcal{C}_{\mathcal{H}}}V) = 0.$$

Summarizing, $T_* : {}_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}} \longrightarrow {}_{\mathbb{K}}\mathfrak{M}$ is a homological and effaceable δ-functor.

For each Hopf bimodule M, our aim now is to endow $T_n(M)$ with a left module structure over $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$. Let us first consider the case $n = 0$. By Proposition 1.10 (3), there is a canonical left $\mathcal{R}_{\mathcal{H}}$-action on $T_0(M)$ such that

$$T_0(M) \cong \mathrm{HH}_0(\mathcal{B}, M)\square_{\mathcal{C}_{\mathcal{H}}}V, \tag{24}$$

the natural $\mathbb{K}$-linear isomorphism constructed in (18), is a homomorphism of $\mathcal{R}_{\mathcal{H}}$-modules. As M is an object in ${}_{\mathcal{Z}_0}\mathfrak{M}^{\mathcal{C}_{\mathcal{H}}}$, by §1.8, it follows that $M\square_{\mathcal{C}_{\mathcal{H}}}V$ is a left $\mathcal{Z}_0$-submodule of $M \otimes V$. In particular this cotensor product is a left $\mathcal{Z}_0$-module. Furthermore, $T_0(M)$ is a quotient $\mathcal{Z}_0$-module of $M\square_{\mathcal{C}_{\mathcal{H}}}V$, as the commutator space $[\mathcal{B}, M\square_{\mathcal{C}_{\mathcal{H}}}V]$ is a $\mathcal{Z}_0$-submodule of $M\square_{\mathcal{C}_{\mathcal{H}}}V$. Obviously, with respect to this module structure, $T_0(M)$ becomes a module over $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$ and the isomorphism in (24) is a map of $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-modules. Since T_* is a homological and effaceable functor, one can proceed as in the proof of Proposition 2.12 to lift the $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-action on $T_0(M)$ to a natural $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-module structure on $T_*(M)$, for every Hopf bimodule M. Again as in the proof of the above mentioned result, we can show that $T_* : {}_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}} \to {}_{\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0}\mathfrak{M}$ is homological and effaceable.

It remains to prove the isomorphisms in (23). Note that the left derived functors of a right exact functor define a homological and effaceable δ-functor. Thus L_*G_V is a homological and effaceable δ-functor. Clearly, $\mathrm{L}_0G_V = \mathrm{HH}_0(\mathcal{B}, -)\square_{\mathcal{C}_{\mathcal{H}}}V$. Hence, by the universal property of homological and effaceable δ-functors, this identity may be lifted to give the first isomorphism in (23). The second isomorphism is obtained in a similar manner, by lifting the natural transformation in (24). $\qquad\square$

2.15. Let $\mathcal{B} \subseteq \mathcal{A}$ be an $\mathcal{H}$-comodule algebra and let M be a Hopf bimodule. Following [Ş2, Theorem 1.3] we regard $C_*(\mathcal{A}, M)$ as a complex in the category $\mathfrak{M}^{\mathcal{C}_{\mathcal{H}}}$ with respect to the coaction that in degree n is given by

$$\varrho_n(M)(m \otimes a^1 \otimes \cdots \otimes a^n) = \sum m_{\langle 0 \rangle} \otimes a^1_{\langle 0 \rangle} \otimes \cdots \otimes a^n_{\langle 0 \rangle} \otimes \pi_{\mathcal{H}}\left(m_{\langle 1 \rangle}a^1_{\langle 1 \rangle}\cdots a^n_{\langle 1 \rangle}\right).$$

Recall that $\pi_{\mathcal{H}}$ denotes the projection of $\mathcal{H}$ onto $\mathcal{C}_{\mathcal{H}}$ and that $\mathcal{Z}_0 = \mathcal{Z}\bigcap\mathcal{B}$. It is not difficult to see that $C_*(\mathcal{A}, M)$ is a complex of left $\mathcal{Z}_0$-modules with respect to the action that in degree n is defined by

$$z \cdot (m \otimes a^1 \otimes \cdots \otimes a^n) = zm \otimes a^1 \otimes \cdots \otimes a^n.$$

In fact, since $\mathcal{Z}_0$ contains only coinvariant elements, it follows that $C_*(\mathcal{A}, M)$ is a complex in ${}_{\mathcal{Z}_0}\mathfrak{M}^{\mathcal{C}_{\mathcal{H}}}$. Therefore, $\mathrm{HH}_n(\mathcal{A}, M)$ is an object in the same category, for

every n. In view of §1.8 it follows that $\mathrm{HH}_*(\mathcal{A}, M)\square_{\mathcal{C}_{\mathcal{H}}}V$ is a left $\mathcal{Z}_0$-module, for any injective left $\mathcal{C}_{\mathcal{H}}$-comodule V. Therefore

$$H_V : {}_{\mathcal{A}}\mathfrak{M}^{\mathcal{H}}_{\mathcal{A}} \to {}_{\mathcal{Z}_0}\mathfrak{M}, \quad H_V(M) := M_{\mathcal{A}}\square_{\mathcal{C}_{\mathcal{H}}}V.$$

is a well defined functor, as by the foregoing remarks $M_{\mathcal{A}} = \mathrm{HH}_0(\mathcal{A}, M)$ is a right $\mathcal{C}_{\mathcal{H}}$-module and $H_V(M)$ is a $\mathcal{Z}_0$-module.

PROPOSITION 2.16. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a faithfully flat $\mathcal{H}$-Galois extension, where $\mathcal{H}$ is a Hopf algebra with bijective antipode. If V is an injective $\mathcal{C}_{\mathcal{H}}$-comodule, then there is an isomorphism of δ-functors*

$$\mathrm{L}_*H_V \cong \mathrm{HH}_*(\mathcal{A}, -)\square_{\mathcal{C}_{\mathcal{H}}}V. \tag{25}$$

Proof. First, let us show that $T_* := \mathrm{HH}_*(\mathcal{A}, -)\square_{\mathcal{C}_{\mathcal{H}}}V$ is a homological and effaceable δ-functor to the category of left $\mathcal{Z}_0$-modules. For a short exact sequence as in (20),

$$0 \longrightarrow C_*(\mathcal{A}, M') \longrightarrow C_*(\mathcal{A}, M) \longrightarrow C_*(\mathcal{A}, M'') \longrightarrow 0$$

is an exact sequence of complexes in ${}_{\mathcal{Z}_0}\mathfrak{M}^{\mathcal{C}_{\mathcal{H}}}$. Therefore, the corresponding long exact sequence in homology lives in the same category. In particular, its connecting maps δ_* are morphisms of $\mathcal{Z}_0$-modules and $\mathcal{C}_{\mathcal{H}}$-comodules, so are $\mathcal{Z}_0$-linear. Since V is injective, the functor $(-)\square_{\mathcal{C}_{\mathcal{H}}}V$ is exact. Thus T_* is a homological functor with connecting maps $\delta_*\square_{\mathcal{C}_{\mathcal{H}}}V$.

By Proposition 2.4, the Hopf bimodule $U := \mathcal{A} \otimes \mathcal{A}$ is a generator. We also have

$$\mathrm{HH}_n(\mathcal{A}, U)\square_{\mathcal{C}_{\mathcal{H}}}V \cong \mathrm{Tor}^{\mathcal{A}^e}_n(\mathcal{A}, \mathcal{A}^e)\square_{\mathcal{C}_{\mathcal{H}}}V = 0.$$

In conclusion T_* is effaceable, as Hochschild homology and the cotensor product commute with direct sums. The isomorphisms in (25) are obtained by lifting the identity $\mathrm{L}_0H_V = T_0$, as in the proof of Proposition 2.14. $\square$

2.17. Since $\mathcal{H}$ is a Hopf algebra, the category of right $\mathcal{H}$-comodules is monoidal, with respect to the tensor product of vector spaces, on which we put the diagonal coaction. More precisely, if V and W are right $\mathcal{H}$-comodules then $\mathcal{H}$ coacts on $V \otimes W$ via the map

$$\rho_{V \otimes W}(v \otimes w) = \sum v_{\langle 0 \rangle} \otimes w_{\langle 0 \rangle} \otimes v_{\langle 1 \rangle}w_{\langle 1 \rangle},$$

where $v \in V$, $w \in W$. Let us denote the right adjoint coaction of $\mathcal{H}$ on itself by $\mathcal{H}^{\mathrm{ad}}$. Recall that the map $\rho_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ that define this coaction is given by

$$\rho_{\mathcal{H}}(h) = \sum h_{(2)} \otimes Sh_{(1)}h_{(3)}.$$

Hence $\mathcal{A} \otimes \mathcal{H}^{\mathrm{ad}}$ is a right $\mathcal{H}$-comodule. Consequently, it is a $\mathcal{C}_{\mathcal{H}}$-comodule via

$$\rho_{\mathcal{A} \otimes \mathcal{H}}(a \otimes h) = \sum a_{\langle 0 \rangle} \otimes h_{(2)} \otimes \pi_{\mathcal{H}}\left(a_{\langle 1 \rangle}Sh_{(1)}h_{(3)}\right). \tag{26}$$

2.18. For a Hopf algebra $\mathcal{H}$ let

$$\mathcal{H}^+ := \ker \varepsilon_{\mathcal{H}} \qquad \text{and} \qquad \mathcal{R}^+_{\mathcal{H}} := \mathcal{H}^+ \cap \mathcal{R}_{\mathcal{H}}.$$

If X is an $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-module, then $\mathcal{R}^+_{\mathcal{H}}X$ is a $\mathcal{Z}_0$-submodule of X, as the corresponding actions of $\mathcal{R}_{\mathcal{H}}$ and $\mathcal{Z}_0$ on X commute. For the same reason, $\mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} X$ is a $\mathcal{Z}_0$-module. Obviously, with respect to these module structures, the canonical isomorphism

$$\mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} X \cong X/\left(\mathcal{R}^+_{\mathcal{H}}X\right)$$

is $\mathcal{Z}_0$-linear. For example, $\mathcal{Z}_0$ acts on $\mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} G_V(M)$ such that, for

$$\zeta = \sum_{i=1}^n [m_i]_{\mathcal{B}} \otimes v_i \tag{27}$$

in $G_V(M)$ and z in $\mathcal{Z}_0$ we have

$$z \cdot (1 \otimes_{\mathcal{R}_{\mathcal{H}}} \zeta) = \sum_{i=1}^n 1 \otimes_{\mathcal{R}_{\mathcal{H}}} [zm_i]_{\mathcal{B}} \otimes v_i.$$

In view of this observation, we shall regard $\mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} G_V$ as a functor from the category of Hopf bimodules to the category of left $\mathcal{Z}_0$-modules.

LEMMA 2.19. *Let $\mathcal{R}$ and $\mathcal{C}$ denote an algebra and a coalgebra, respectively, over a field $\mathbb{K}$. If $(M, \rho_M) \in {}_{\mathcal{R}}\mathfrak{M}^{\mathcal{C}}$ then there is a unique morphism of δ-functors*

$$\rho_*(-) : \mathrm{Tor}_*^{\mathcal{R}}(-, M) \to \mathrm{Tor}_*^{\mathcal{R}}(-, M) \otimes \mathcal{C}$$

that lifts $\rho_0(-) := (-) \otimes_{\mathcal{R}} \rho_M$ and defines a $\mathcal{C}$-coaction on $\mathrm{Tor}_*^{\mathcal{R}}(-, M)$.

Proof. Obviously, $\mathrm{Tor}_*^{\mathcal{R}}(-, M)$ and $\mathrm{Tor}_*^{\mathcal{R}}(-, M) \otimes \mathcal{C}$ are homological and effaceable δ-functors, which are defined on the category of right $\mathcal{R}$-modules. In degree zero, $\rho_0(-) := (-) \otimes_{\mathcal{R}} \rho_M$ defines a $\mathcal{C}$-comodule structure on $\mathrm{Tor}_0^{\mathcal{R}}(-, M) = (-) \otimes_{\mathcal{R}} \mathcal{C}$. By the universal property, there is a morphism of δ-functors

$$\rho_*(-) : \mathrm{Tor}_n^{\mathcal{R}}(-, M) \to \mathrm{Tor}_n^{\mathcal{R}}(-, M) \otimes \mathcal{C}$$

that lifts ρ_0. We want to prove that $\rho_*(-)$ defines a coaction on $\mathrm{Tor}_*^{\mathcal{R}}(-, M)$. We have already remarked that this property holds in degree zero, so

$$\bigl(\rho_0(-) \otimes \mathcal{C}\bigr) \circ \rho_0(-) = (- \otimes \Delta_{\mathcal{C}}) \circ \rho_0(-). \tag{28}$$

We need a similar identity for $\rho_n(-)$. Note that $\mathrm{Tor}_*^{\mathcal{R}}(-, M) \otimes \mathcal{C} \otimes \mathcal{C}$ is also a homological and effaceable δ-functor. Clearly, $\bigl(\rho_*(-) \otimes \mathcal{C}\bigr) \circ \rho_*(-)$ and $(- \otimes \Delta_{\mathcal{C}}) \circ \rho_*(-)$ lift the natural transformations in the left and respectively right hand sides of (28). Hence, by the uniqueness of the lifting, these morphisms of δ-functors are equal (see the universal property in §1.9). $\square$

REMARK 2.20. In view of the previous lemma, $\rho_*(-)$ is a homomorphism of δ-functors. Hence, for an exact sequence of right $\mathcal{R}$-modules as in (15) and an object M in ${}_{\mathcal{R}}\mathfrak{M}^{\mathcal{C}}$, the connecting maps

$$\delta_* : \mathrm{Tor}_*^{\mathcal{R}}(X'', M) \to \mathrm{Tor}_{*-1}^{\mathcal{R}}(X', M)$$

are morphisms of $\mathcal{C}$-comodules.

LEMMA 2.21. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a faithfully flat $\mathcal{H}$-Galois extension over a Hopf algebra such that $S_{\mathcal{H}}^2 = \mathrm{Id}_{\mathcal{H}}$. Let V be an injective $\mathcal{C}_{\mathcal{H}}$-comodule and set $U := \mathcal{A} \otimes \mathcal{A}$.*

(1) *Let G_V and H_V be the functors defined respectively in §2.13 and §2.15. For every Hopf bimodule M and ζ as in (27), the formula*

$$\phi(M)(1 \otimes \zeta) = \sum_{i=1}^n [m_i]_{\mathcal{A}} \otimes v_i$$

defines a natural transformation $\phi(-) : \mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} G_V(-) \longrightarrow H_V(-)$.

(2) *The algebra $\mathcal{R}_{\mathcal{H}}$ acts on $\mathcal{A} \otimes \mathcal{H}$ via the multiplication in $\mathcal{H}$ so that $\mathcal{A} \otimes \mathcal{H}$ is an object in ${}_{\mathcal{R}_{\mathcal{H}}}\mathfrak{M}^{\mathcal{C}_{\mathcal{H}}}$ with respect to the coaction (26).*

(3) *The $\mathcal{C}_{\mathcal{H}}$-coaction on $\mathcal{A} \otimes \mathcal{H}$ induces a comodule structure on $\mathrm{Tor}_*^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, \mathcal{A} \otimes \mathcal{H})$.*

(4) *We assume, in addition, that* $\mathcal{R}_{\mathcal{H}}^{+}\mathcal{H} = \mathcal{H}^{+}$ *and* $\operatorname{Tor}_{n}^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, \mathcal{H}) = 0$, *for every* $n > 0$. *Then* $\phi(U)$ *is an isomorphism and* $\operatorname{Tor}_{n}^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, G_{V}(U)) = 0$, *for* $n > 0$.

Proof. (1) Let M be a Hopf bimodule and let $p(M) : M_{\mathcal{B}} \to M_{\mathcal{A}}$ denote the canonical projection. Clearly, $p(M)$ is a natural morphism of $\mathcal{C}_{\mathcal{H}}$-modules. To simplify the notation, set $p := p(M)$. By the foregoing, $f := p \,\square_{\mathcal{C}_{\mathcal{H}}} V$ is well-defined. Furthermore, by [JŞ, Proposition 2.6],

$$\sum \kappa^{1}(r)\kappa^{2}(r) = \varepsilon(r).$$

Let ζ be an element in $M_{\mathcal{B}}\square_{\mathcal{C}_{\mathcal{H}}}V$ satisfying relation (27). Since $[am]_{\mathcal{A}} = [ma]_{\mathcal{A}}$, for every $r \in \mathcal{R}_{\mathcal{H}}$, a straightforward computation yields

$$f(r \cdot \zeta) = \sum_{i=1}^{n}[\kappa^{2}(r)m_{i}\kappa^{1}(r)]_{\mathcal{A}} \otimes v_{i} = \varepsilon(r)f(\zeta).$$

Thus, there exists a natural map $\phi(M) : \mathbb{K} \otimes_{\mathcal{R}_{H}} G_{V}(M) \to H_{V}(M)$ of $\mathcal{Z}_{0}$-modules, which is uniquely defined such that

$$\phi(M)(1_{\mathbb{K}} \otimes_{\mathcal{R}_{\mathcal{H}}} \zeta) = f(\zeta).$$

(2) We regard $\mathcal{A} \otimes \mathcal{H}$ as a left $\mathcal{R}_{\mathcal{H}}$-module via the multiplication in $\mathcal{H}$. Let us prove that $\rho_{\mathcal{A}\otimes\mathcal{H}}$ is a morphism of $\mathcal{R}_{\mathcal{H}}$-modules. We pick up $a \in \mathcal{A}$, $h \in \mathcal{H}$ and $r \in \mathcal{R}_{\mathcal{H}}$. Thus

$$
\begin{aligned}
\rho_{\mathcal{A}\otimes\mathcal{H}}(a \otimes rh) &= \sum a_{\langle 0\rangle} \otimes r_{(2)}h_{(2)} \otimes \pi_{\mathcal{H}}\left(a_{\langle 1\rangle}S_{\mathcal{H}}(r_{(1)}h_{(1)})r_{(3)}h_{(3)}\right) \\
&= \sum a_{\langle 0\rangle} \otimes r_{(2)}h_{(2)} \otimes \pi_{\mathcal{H}}\left(a_{\langle 1\rangle}S_{\mathcal{H}}h_{(1)}S_{\mathcal{H}}r_{(1)}r_{(3)}h_{(3)}\right) \\
&= \sum a_{\langle 0\rangle} \otimes r_{(3)}h_{(2)} \otimes \pi_{\mathcal{H}}\left(a_{\langle 1\rangle}S_{\mathcal{H}}h_{(1)}S_{\mathcal{H}}r_{(2)}r_{(1)}h_{(3)}\right) \\
&= \sum a_{\langle 0\rangle} \otimes rh_{(2)} \otimes \pi_{\mathcal{H}}\left(a_{\langle 1\rangle}S_{\mathcal{H}}h_{(1)}h_{(3)}\right) \\
&= r \cdot \rho_{\mathcal{A}\otimes\mathcal{H}}(a \otimes h).
\end{aligned}
$$

Note that the third equality follows by (13), while in the fourth one we used (17).

(3) This part is a direct application of Lemma 2.19.

(4) Let $\lambda := \beta \circ \eta$, where β is the canonical map in the definition of Hopf-Galois extensions and η is the following $\mathbb{K}$-linear isomorphism

$$\eta : (\mathcal{A} \otimes \mathcal{A})_{\mathcal{B}} \to \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}, \quad \eta\left([a \otimes x]_{\mathcal{B}}\right) = x \otimes_{\mathcal{B}} a.$$

By the definition of β and η we can easily show that

$$\lambda\left([a \otimes x]_{\mathcal{B}}\right) = \sum xa_{\langle 0\rangle} \otimes a_{\langle 1\rangle}.$$

We claim that λ is an isomorphism in $_{\mathcal{R}_{\mathcal{H}}}\mathfrak{M}^{\mathcal{C}_{\mathcal{H}}}$. Obviously, λ is bijective as β and η are so. If $r \in \mathcal{R}_{\mathcal{H}}$ and $a, x \in \mathcal{A}$ then

$$
\begin{aligned}
\lambda(r \cdot [a \otimes x]_{\mathcal{B}}) &= \lambda\left(\sum[\kappa^{2}(r)a \otimes x\kappa^{1}(r)]_{B}\right) \\
&= \sum x\kappa^{1}(r)\kappa^{2}(r)_{\langle 0\rangle}a_{\langle 0\rangle} \otimes \kappa^{2}(r)_{\langle 1\rangle}a_{\langle 1\rangle} \\
&= \sum xa_{\langle 0\rangle} \otimes ra_{\langle 1\rangle},
\end{aligned}
$$

where for the last equality we used (11). Thus λ is a morphism of $\mathcal{R}_{\mathcal{H}}$-modules. Let ρ denote the coaction of $\mathcal{C}_{\mathcal{H}}$ on $(\mathcal{A} \otimes \mathcal{A})_{\mathcal{B}}$. Hence, by the definition of ρ and the fact that $\pi_{\mathcal{H}}$ is a trace map, we get

$$
\begin{aligned}
(\lambda \otimes \mathcal{H}) \circ \rho([a \otimes x]_{\mathcal{B}}) &= \sum \lambda\left([a_{\langle 0\rangle} \otimes x_{\langle 0\rangle}]_{\mathcal{B}}\right) \otimes \pi_{\mathcal{H}}(a_{\langle 1\rangle}x_{\langle 1\rangle}) \\
&= \sum x_{\langle 0\rangle}a_{\langle 0\rangle} \otimes a_{\langle 1\rangle} \otimes \pi_{\mathcal{H}}(x_{\langle 1\rangle}a_{\langle 2\rangle}).
\end{aligned}
$$

On the other hand, by (26), it follows

$$\begin{aligned}
\rho_{\mathcal{A} \otimes \mathcal{H}} \circ \lambda([a \otimes x]_{\mathcal{B}}) &= \rho_{\mathcal{A} \otimes \mathcal{H}} \left(\sum x a_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle} \right) \\
&= \sum (x a_{\langle 0 \rangle})_{\langle 0 \rangle} \otimes (a_{\langle 1 \rangle})_{(2)} \otimes \pi_{\mathcal{H}} \left((x a_{\langle 0 \rangle})_{\langle 1 \rangle} S_{\mathcal{H}}(a_{\langle 1 \rangle_{(1)}}) a_{\langle 1 \rangle_{(3)}} \right) \\
&= \sum x_{\langle 0 \rangle} a_{\langle 0 \rangle} \otimes a_{\langle 3 \rangle} \otimes \pi_{\mathcal{H}}(x_{\langle 1 \rangle} a_{\langle 1 \rangle} S_{\mathcal{H}} a_{\langle 2 \rangle} a_{\langle 4 \rangle}) \\
&= \sum x_{\langle 0 \rangle} a_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle} \otimes \pi_{\mathcal{H}}(x_{\langle 1 \rangle} a_{\langle 2 \rangle}).
\end{aligned}$$

Summarizing, the computation above shows us that λ is a morphism of $\mathcal{C}_{\mathcal{H}}$-comodules too. Furthermore, for a right $\mathcal{R}_{\mathcal{H}}$-module N, we get

$$N \otimes_{\mathcal{R}_{\mathcal{H}}} G_V(U) \cong N \otimes_{\mathcal{R}_{\mathcal{H}}} [(\mathcal{A} \otimes \mathcal{H}) \square_{\mathcal{C}_{\mathcal{H}}} V] \cong [N \otimes_{\mathcal{R}_{\mathcal{H}}} (\mathcal{A} \otimes \mathcal{H})] \square_{\mathcal{C}_{\mathcal{H}}} V,$$

where the first isomorphism is defined by $N \otimes_{\mathcal{R}_{\mathcal{H}}} (\lambda \square_{\mathcal{C}_{\mathcal{H}}} V)$ and the second one comes from the commutation of the tensor product and the cotensor product. We have obtained a natural isomorphism

$$\nu(N) : N \otimes_{\mathcal{R}_{\mathcal{H}}} G_V(U) \to [\mathcal{A} \otimes (N \otimes_{\mathcal{R}_{\mathcal{H}}} \mathcal{H})] \square_{\mathcal{C}_{\mathcal{H}}} V$$

given, for $z := \sum_{i=1}^n [a^i \otimes b^i]_{\mathcal{B}} \otimes v^i$ in $G_V(U)$ and $x \in N$, by

$$\nu(N)(x \otimes_{\mathcal{R}_{\mathcal{H}}} z) = \sum_{i=1}^k \sum b^i a^i_{\langle 0 \rangle} \otimes (x \otimes_{\mathcal{R}_{\mathcal{H}}} a^i_{\langle 1 \rangle}) \otimes v^i.$$

Let us prove that $\phi(U)$ is an isomorphism. Since $\mathcal{H}/\mathcal{R}_{\mathcal{H}}^+ \mathcal{H} \cong \mathbb{K}$,

$$\mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} \mathcal{H} \cong \mathcal{R}_{\mathcal{H}}/\mathcal{R}_{\mathcal{H}}^+ \otimes_{\mathcal{R}_{\mathcal{H}}} \mathcal{H} \cong \mathcal{H}/\mathcal{R}_{\mathcal{H}}^+ \mathcal{H} \cong \mathbb{K}.$$

Note that this isomorphism maps $1 \otimes_{\mathcal{R}_{\mathcal{H}}} h$ to $\varepsilon(h)$. Let γ be the composition of the isomorphism $[\mathcal{A} \otimes (\mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} \mathcal{H})] \square_{\mathcal{C}_{\mathcal{H}}} V \cong \mathcal{A} \square_{\mathcal{C}_{\mathcal{H}}} V$ and $\nu(\mathbb{K})$. Then

$$\gamma(1 \otimes_{\mathcal{R}_{\mathcal{H}}} z) = \sum_{i=1}^k \sum b^i a^i \otimes v^i,$$

where $z \in G_V(U)$ is given by the same formula as above. Furthermore, the multiplication in $\mathcal{A}$ induces an isomorphism of $\mathcal{C}_{\mathcal{H}}$-comodules

$$\mu : U_{\mathcal{A}} \longrightarrow \mathcal{A}, \qquad \mu([a' \otimes a'']_{\mathcal{A}}) = a'' a'.$$

It is easy to see that $\phi(U) = (\mu \square_{\mathcal{C}_{\mathcal{H}}} V) \circ \gamma$, so $\phi(U)$ is an isomorphism.

Let P_* be a resolution of $\mathbb{K}$ in $\mathfrak{M}_{\mathcal{R}_{\mathcal{H}}}$. The natural transformation ν yields isomorphisms

$$P_* \otimes_{\mathcal{R}_{\mathcal{H}}} G_V(U) \cong [\mathcal{A} \otimes (P_* \otimes_{\mathcal{R}_{\mathcal{H}}} \mathcal{H})] \square_{\mathcal{C}_{\mathcal{H}}} V.$$

Since $\mathcal{A} \otimes (-)$ and $(-) \square_{\mathcal{C}_{\mathcal{H}}} V$ are exact functors it follows

$$\mathrm{Tor}_n^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, G_V(U)) \simeq [\mathcal{A} \otimes \mathrm{Tor}_n^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, \mathcal{H})] \square_{\mathcal{C}_{\mathcal{H}}} V.$$

Hence the lemma is completely proven as $\mathrm{Tor}_n^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, \mathcal{H}) = 0$, for every $n > 0$. $\square$

DEFINITION 2.22. We say that a Hopf algebra $\mathcal{H}$ *has enough cocommutative elements* if $\mathcal{R}_{\mathcal{H}}^+ \mathcal{H} = \mathcal{H}^+$.

THEOREM 2.23. *Let $\mathcal{H}$ be a Hopf algebra such that $S_{\mathcal{H}}^2 = \mathrm{Id}_{\mathcal{H}}$. We assume that $\mathcal{H}$ has enough cocommutative elements and $\mathrm{Tor}_*^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, \mathcal{H}) = 0$. If $\mathcal{B} \subseteq \mathcal{A}$ is a faithfully flat $\mathcal{H}$-Galois extension and V is an injective left $\mathcal{C}_{\mathcal{H}}$-comodule then, for every Hopf bimodule M, there is a spectral sequence in the category $_{\mathcal{Z}_0}\mathfrak{M}$*

$$\mathrm{Tor}_p^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, \mathrm{HH}_q(\mathcal{B}, M \square_{\mathcal{C}_{\mathcal{H}}} V)) \Longrightarrow \mathrm{HH}_{p+q}(\mathcal{A}, M) \square_{\mathcal{C}_{\mathcal{H}}} V. \tag{29}$$

Proof. We know that $U := \mathcal{A} \otimes \mathcal{A}$ is a generator in $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}$. In view of Lemma 2.21, one can apply Proposition 2.3 to the following categories:

$$\mathfrak{A} := {_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{\mathcal{H}}}, \qquad \mathfrak{B} := {_{\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0}\mathfrak{M}}, \qquad \mathfrak{C} := {_{\mathcal{Z}_0}\mathfrak{M}}.$$

The functors F, G_V and H are given by

$$F := \mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} (-), \qquad G_V := (-)_{\mathcal{B}}\square_{\mathcal{C}_{\mathcal{H}}} V, \qquad H_V := (-)_{\mathcal{A}}\square_{\mathcal{C}_{\mathcal{H}}} V$$

and the natural transformation $\phi : F \circ G_V \to H_V$ is defined in Lemma 2.21 (1). To compute the left derived functors of G_V and H_V we use Propositions 2.14 and 2.16. Since any projective $\mathcal{R}_{\mathcal{H}} \otimes \mathcal{Z}_0$-module is also projective as an $\mathcal{R}_{\mathcal{H}}$-module, it follows that $\mathrm{L}_n F \cong \mathrm{Tor}_n^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, -)$. $\qquad\square$

PROPOSITION 2.24. *Let $\mathcal{H}$ be a finite-dimensional Hopf algebra over a field $\mathbb{K}$ of characteristic zero such that $S_{\mathcal{H}}^2 = \mathrm{Id}_{\mathcal{H}}$. Then $\mathcal{R}_{\mathcal{H}}$ is semisimple and $\mathcal{C}_{\mathcal{H}}$ is cosemisimple.*

Proof. We first prove that $\mathcal{R}_{\mathcal{H}}$ is semisimple in the case when $\mathbb{K}$ is algebraically closed. By Larson-Radford Theorem [DNR, Theorem 7.4.6], it follows that $\mathcal{H}$ is semisimple and cosemisimple. We claim that, in this particular case, $\mathcal{R}_{\mathcal{H}^*}$ equals the $\mathbb{K}$-subalgebra $C_{\mathbb{K}}(\mathcal{H})$ of $\mathcal{H}^*$, which is generated by the set of characters of $\mathcal{H}$. For the definition and properties of characters of a semisimple Hopf algebra, the reader is referred to [DNR, Section 7.5]. Recall that an element $\alpha \in \mathcal{H}^*$ is said to be a trace map on $\mathcal{H}$ if and only if α vanishes on the space of commutators $[\mathcal{H}, \mathcal{H}]$. Let us show that $\mathcal{R}_{\mathcal{H}^*}$ equals the space of all trace maps on $\mathcal{H}$. By the definition of comultiplication of $\mathcal{H}^*$,

$$\Delta(\alpha) = \sum \alpha_{(1)} \otimes \alpha_{(2)}$$

if and only if $\alpha(xy) = \sum \alpha_{(1)}(x)\alpha_{(2)}(y)$, for all $x, y \in \mathcal{H}$. On the other hand, $\alpha \in \mathcal{R}_{\mathcal{H}^*}$ if and only if $\Delta(\alpha) = \sum \alpha_{(2)} \otimes \alpha_{(1)}$. Therefore, for $\alpha \in \mathcal{R}_{\mathcal{H}^*}$, we get

$$\alpha(xy) = \sum \alpha_{(2)}(x)\alpha_{(1)}(y) = \sum \alpha_{(1)}(y)\alpha_{(2)}(x) = \alpha(yx),$$

so α is a trace map. The other implication can be proved similarly. We can now show that $\mathcal{R}_{\mathcal{H}^*} = C_{\mathbb{K}}(\mathcal{H})$. By definition, a character is a trace map, so $C_{\mathbb{K}}(\mathcal{H})$ is a subspace of $\mathcal{R}_{\mathcal{H}^*}$. Therefore, it is enough to show that $\dim \mathcal{R}_{\mathcal{H}^*} \leq \dim C_{\mathbb{K}}(\mathcal{H})$. As the base field is algebraically closed, $\mathcal{H} \cong \prod_{i=1}^n M_{d_i}(\mathbb{K})$. For every $i = 1, \ldots, n$, let V_i be a simple left $\mathcal{H}$-module associated to the block $M_{d_i}(\mathbb{K})$ and let χ_i denote the irreducible character corresponding to V_i. By [DNR, Proposition 7.5.7], $\chi_1, \ldots, \chi_n$ are linearly independent over $\mathbb{K}$, as elements in $\mathcal{H}^*$. On the other hand, using the canonical basis $\{E_{ip_iq_i} \mid i = 1, \ldots, n, \ p_i, q_i = 1, \ldots, d_i\}$ on $\prod_{i=1}^n M_{d_i}(\mathbb{K})$, one can show that α is a trace map if and only if there are $a_1, \ldots, a_n$ in $\mathbb{K}$ such that

$$\alpha(E_{ip_iq_i}) = \begin{cases} 0, & \text{if } p_i \neq q_i, \\ a_i, & \text{if } p_i = q_i. \end{cases}$$

Hence, $\dim \mathcal{R}_{\mathcal{H}^*} = n \leq \dim C_{\mathbb{K}}(\mathcal{H})$. To deduce that $\mathcal{R}_{\mathcal{H}^*}$ is semisimple, we now use [DNR, Theorem 7.5.12] and the fact that $C_{\mathbb{K}}(\mathcal{H}) = \mathbb{K} \otimes_{\mathbb{Q}} C_{\mathbb{Q}}(\mathcal{H})$. We have already remarked that $\mathcal{H}$ is cosemisimple too. Thus, $\mathcal{R}_{\mathcal{H}} \cong \mathcal{R}_{\mathcal{H}^{**}}$ is also semisimple.

We now assume that $\mathbb{K}$ is an arbitrary field of characteristic zero. Let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$ and let $\overline{\mathcal{H}} := \overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathcal{H}$. We claim that $\overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathcal{R}_{\mathcal{H}} = \mathcal{R}_{\overline{\mathcal{H}}}$. Indeed,

let $\{\alpha_i \mid i \in I\}$ be a basis of $\overline{\mathbb{K}}$ as a $\mathbb{K}$-vector space and $z = \sum_{i \in I} \alpha^i \otimes h^i \in \overline{\mathcal{H}}$. By the definition of the comultiplication of $\overline{\mathcal{H}}$, z belongs to $\mathcal{R}_{\overline{\mathcal{H}}}$ if and only if

$$\sum_{i \in I} (\alpha^i \otimes h^i_{(1)}) \otimes_{\overline{\mathbb{K}}} (1 \otimes h^i_{(2)}) = \sum_{i \in I} (1 \otimes h^i_{(2)}) \otimes_{\overline{\mathbb{K}}} (\alpha^i \otimes h^i_{(1)}).$$

Thus $\sum_{i \in I} \alpha^i \otimes h^i_{(1)} \otimes h^i_{(2)} = \sum_{i \in I} \alpha^i \otimes h^i_{(2)} \otimes h^i_{(1)}$. Since the elements α_i are linearly independent over $\mathbb{K}$ it results that $z \in \mathcal{R}_{\overline{\mathcal{H}}}$ if and only if each h_i is an element in $\mathcal{R}_{\mathcal{H}}$. Consequently, the claimed equality is proven.

Obviously, $S_{\overline{\mathcal{H}}}^2 = \mathrm{Id}_{\overline{\mathcal{H}}}$. Since $\overline{\mathcal{H}}$ is a Hopf algebra over an algebraically closed field, it follows that $\mathcal{R}_{\overline{\mathcal{H}}}$ is semisimple. Let J be the Jacobson radical of $\mathcal{R}_{\mathcal{H}}$ which is a finite-dimensional algebra. Thus J is a nilpotent ideal. Clearly $\overline{\mathbb{K}} \otimes_{\mathbb{K}} J$ is a nilpotent ideal in $\overline{\mathbb{K}} \otimes_{\mathbb{K}} \mathcal{R}_{\mathcal{H}} \cong \mathcal{R}_{\overline{\mathcal{H}}}$, so it is contained in the Jacobson radical of $\mathcal{R}_{\overline{\mathcal{H}}}$. We deduce that $\overline{\mathbb{K}} \otimes_{\mathbb{K}} J = 0$. Thus $J = 0$, so $\mathcal{R}_{\mathcal{H}}$ is semisimple, being finite-dimensional.

It remains to prove that $\mathcal{C}_{\mathcal{H}}$ is cosemisimple. The dual algebra $\mathcal{C}_{\mathcal{H}}^*$ is isomorphic to the subalgebra of trace maps on $\mathcal{H}$, i.e. $\mathcal{C}_{\mathcal{H}}^* \cong \mathcal{R}_{\mathcal{H}^*}$. As $\mathcal{H}$ is cosemisimple, we have already seen that $\mathcal{C}_{\mathcal{H}}^* \cong \mathcal{R}_{\mathcal{H}^*}$ is semisimple. Hence $\mathcal{C}_{\mathcal{H}}$ is cosemisimple. $\square$

THEOREM 2.25. *Let $\mathcal{B} \subseteq \mathcal{A}$ be an $\mathcal{H}$-Galois extension, where $\mathcal{H}$ is a Hopf algebra of finite dimension over a field of characteristic zero. If $\mathcal{H}$ has enough cocommutative elements and $S_{\mathcal{H}}^2 = \mathrm{Id}_{\mathcal{H}}$ then, for every Hopf bimodule M and every left $\mathcal{C}_{\mathcal{H}}$-comodule V, there is isomorphisms of $\mathcal{Z}_0$-modules*

$$\mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} \mathrm{HH}_n(\mathcal{B}, M \square_{C_{\mathcal{H}}} V) \cong \mathrm{HH}_n(\mathcal{A}, M) \square_{C_{\mathcal{H}}} V.$$

Proof. By the proof of the previous proposition, $\mathcal{H}$ is cosemisimple. Thus $\mathcal{A}$ is injective as an $\mathcal{H}$-comodule, so the extension $\mathcal{B} \subseteq \mathcal{A}$ is faithfully flat, cf. [SS, Theorem 4.10]. In view of the same proposition V is injective, as $\mathcal{C}_{\mathcal{H}}$ is cosemisimple, and $\mathbb{K}$ is projective as a right $\mathcal{R}_{\mathcal{H}}$-module. Therefore, under the assumptions of the theorem, the spectral sequence (29) exists and collapses. The edge maps of this spectral sequence yields the required isomorphisms. $\square$

For another application of Theorem 2.23, let us take the Hopf algebra $\mathcal{H}$ to be cocommutative. In this case $\mathcal{R}_{\mathcal{H}} = \mathcal{H}$, so the assumptions on $\mathcal{H}$ are trivially satisfied. We obtain the spectral sequence from the following corollary. Note that a related result can be found in [Ş2, Theorem 3.1], where the extension $\mathcal{B} \subseteq \mathcal{A}$ is not necessarily faithfully flat but V is just a subcoalgebra of $\mathcal{C}_{\mathcal{H}}$ which is injective in $^{\mathcal{C}_{\mathcal{H}}}\mathfrak{M}$.

COROLLARY 2.26. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a faithfully flat $\mathcal{H}$-Galois extension, with $\mathcal{H}$ a cocommutative Hopf algebra. If V is an injective right $\mathcal{C}_{\mathcal{H}}$-comodule and M is a Hopf bimodule then there exists a spectral sequence in the category $_{\mathcal{Z}_0}\mathfrak{M}$*

$$\mathrm{Tor}_p^{\mathcal{H}}(\mathbb{K}, \mathrm{HH}_q(\mathcal{B}, M \square_{C_{\mathcal{H}}} V)) \Longrightarrow \mathrm{HH}_{p+q}(\mathcal{A}, M) \square_{C_{\mathcal{H}}} V.$$

REMARK 2.27. By taking $V := \mathcal{C}_{\mathcal{H}}$ in the above corollary we obtain (only in the case of cocommutative Hopf algebras) the spectral sequence [Ş1, Theorem 4.5].

Let now consider the case when the Hopf algebra $\mathcal{H}$ is the group algebra $\mathbb{K}G$ of an arbitrary group G. By [Mo, Theorem 8.1.7], an extension $\mathcal{B} \subseteq \mathcal{A}$ is $\mathbb{K}G$-Galois if

and only if $\mathcal{A}$ is G-strongly graded and $\mathcal{B}$ is its homogeneous component of degree one, i.e. $\mathcal{A}$ is a direct sum of linear subspaces $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ such that $\mathcal{A}_1 = \mathcal{B}$ and

$$\mathcal{A}_g \mathcal{A}_h = \mathcal{A}_{gh},$$

for any g, h in G. A strongly graded algebra, i.e. a $\mathbb{K}G$-Galois extension $\mathcal{B} \subseteq \mathcal{A}$, is always faithfully flat, as $\mathbb{K}G$ is cosemisimple. Furthermore, the coalgebra $\mathcal{C}_{\mathbb{K}G}$ is cosemisimple and pointed, cf. [Ș2, Exemple 1.2 (a)]. A direct application of the preceding corollary, for $V := \mathcal{C}_{\mathbb{K}G}$, yields [Lo1, Theorem 2.5 (a)].

Furthermore, a $\mathbb{K}G$-comodule is a vector space M together with a decomposition as a direct sum of subspaces

$$M = \bigoplus_{g \in G} M_g.$$

Hence, an $\mathcal{A}$-bimodule M is a Hopf bimodule if and only if the above decomposition satisfies, for any g and h in G, the following relations

$$\mathcal{A}_h M_g \subseteq M_{hg} \qquad \text{and} \qquad M_g \mathcal{A}_h \subseteq M_{gh}.$$

Thus, to give a Hopf bimodule is equivalent to give a G-graded $\mathcal{A}$-bimodule.

The coalgebra $\mathcal{C}_{\mathbb{K}G}$ is cosemisimple and pointed, cf. [Ș2, Exemple 1.2 (a)]. Recall that on $\mathcal{C}_{\mathbb{K}G}$ there is a canonical basis $\{e_\sigma \mid \sigma \in T(G)\}$, where $T(G)$ is the set of conjugacy classes in G and each e_σ is a group-like element. A left (or right) $\mathcal{C}_{\mathbb{K}G}$ comodule structure $\rho_V : V \longrightarrow V \otimes \mathbb{K}G$ on a given vector space V is uniquely defined by a decomposition of V as a direct sum

$$V = \bigoplus_{\sigma \in T(G)} V_\sigma.$$

Note that the subspace V_σ is given by

$$V_\sigma = \{v \in V \mid \rho_V(v) = e_\sigma \otimes v\}.$$

We shall say that V_σ is the homogeneous component of V of degree σ.

We now fix a conjugacy class σ in G and we put $V := \mathbb{K}e_\sigma$. Since e_σ is a group-like element, V is a left $\mathcal{C}_{\mathbb{K}G}$-subcomodule of $\mathcal{C}_{\mathbb{K}G}$ and $V_\sigma = V$. Let M be a G-graded bimodule and $M_\sigma := \bigoplus_{g \in \sigma} M_g$. Thus

$$M \square_{\mathcal{C}_\mathcal{H}} V = M_\sigma.$$

Therefore, if G_V and H_V are the functors in the proof of Theorem 2.23 then

$$G_V(M) \cong \mathcal{B} \otimes_{\mathcal{B}^e} M_\sigma, \qquad\qquad H_V(M) := (M_\mathcal{A})_\sigma.$$

Let us notice that $M_\mathcal{A}$ is a right $\mathcal{C}_{\mathbb{K}G}$-comodule, so it makes sense to speak about $(M_\mathcal{A})_\sigma$. More generally, the coaction of $\mathcal{C}_{\mathbb{K}G}$ on the Hochschild homology of $\mathcal{A}$ with coefficients in M induces a decomposition of $\mathrm{HH}_*(\mathcal{A}, M)$ as a direct sum of its homogeneous components $\mathrm{HH}_*(\mathcal{A}, M)_\sigma$, for σ arbitrary in $T(G)$.

As an application of Theorem 2.23 we now get the following result, that was also proved in [Lo1, Theorem 2.5 (b)] by a different method.

COROLLARY 2.28. *If $\mathcal{B} \subseteq \mathcal{A}$ is a strongly G-graded algebra then, for any graded $\mathcal{A}$-bimodule M and $\sigma \in T(G)$, there is a natural spectral sequence in $_{\mathcal{Z}_0}\mathfrak{M}$*

$$\mathrm{H}_p(G, \mathrm{HH}_q(\mathcal{B}, M_\sigma)) \implies \mathrm{HH}_{p+q}(\mathcal{A}, M)_\sigma. \qquad (30)$$

Proof. Group homology $\mathrm{H}_*(G, X)$ and $\mathrm{Tor}_*^{\mathbb{K}G}(\mathbb{K}, X)$ are equal for any $\mathbb{K}G$-module X, cf. [We, Theorem 3.6.2]. $\qquad\square$

REMARK 2.29. We keep the notation from the previous corollary. Let us pick up an element g in σ and denote the centralizer of g in G by $C_G(g)$. One can show that

$$\mathrm{HH}_q(\mathcal{B}, M_\sigma)) \cong \mathbb{K}G \otimes_{\mathbb{K}C_G(g)} \mathrm{HH}_q(\mathcal{B}, M_g).$$

Thus, by Shapiro's Lemma, the terms in the second page of the spectral sequence (30) are isomorphic to

$$E^2_{p,q} = \mathrm{H}_p(C_G(g), \mathrm{HH}_q(\mathcal{B}, M_g)).$$

For details the reader is referred to [Lo1, p. 504].

Now we are going to investigate the case when $\mathcal{H}$ is commutative but not necessarily cocommutative. Thus $\mathcal{C}_\mathcal{H} = \mathcal{H}$. Our first aim is to show that we can drop the assumptions on $\mathcal{R}_\mathcal{H}$ in Theorem 2.23. To this end, we need the following.

LEMMA 2.30. *Let G be a finite group. If $\mathcal{H} := (\mathbb{K}G)^*$ then $\mathcal{H}$ has enough cocommutative elements and $\mathcal{R}_\mathcal{H}$ is semisimple.*

Proof. Let $\{p_x \mid x \in G\}$ be the dual basis of the canonical basis on $\mathbb{K}G$. By definition, the coalgebra structure on $\mathcal{H}$ is given by

$$\Delta(p_x) = \sum_{g \in G} p_{xg^{-1}} \otimes p_g \qquad \text{and} \qquad \varepsilon(p_x) = \delta_{x,1}.$$

Thus, an element $z = \sum_{x \in G} a_x p_x$ belongs to $\mathcal{R}_\mathcal{H}$ if and only if

$$\sum_{x,g \in G} a_x p_{xg^{-1}} \otimes p_g = \sum_{x,g \in G} a_x p_g \otimes p_{xg^{-1}}.$$

Since $\{p_x \otimes p_y \mid x, y \in G\}$ is a basis on $\mathcal{H} \otimes \mathcal{H}$, we deduce that $a_{xg} = a_{gx}$. Therefore, if $\sigma \in T(G)$ then there is a_σ in $\mathbb{K}$ such that $a_g = a_\sigma$, for all $g \in \sigma$. It follows

$$z = \sum_{\sigma \in T(G)} a_\sigma p_\sigma,$$

where $p_\sigma := \sum_{x \in \sigma} p_x$. In conclusion, $\mathcal{R}_\mathcal{H}$ is the $\mathbb{K}$-linear subspace generated by all p_σ, with $\sigma \in T(G)$. On the other hand, $p_x p_y = \delta_{x,y} p_x$, for arbitrary $x, y \in G$. Thus, if $x \in G$ and $x \neq 1$, then

$$p_x = p_\sigma p_x,$$

where σ denotes the conjugacy class of x. Hence the relation $\mathcal{R}_\mathcal{H}^+ \mathcal{H} = \mathcal{H}^+$ is proven. To conclude the proof we remark that

$$p_\sigma p_\tau = \delta_{\sigma,\tau} p_\sigma \qquad \text{and} \qquad \sum_{\sigma \in T(G)} p_\sigma = 1_{\mathcal{R}_\mathcal{H}},$$

so $\mathcal{R}_\mathcal{H}$ is semisimple. In fact, the above relations shows us that $\mathcal{R}_\mathcal{H} \simeq \mathbb{K}^{\#T(G)}$. $\square$

PROPOSITION 2.31. *Let $\mathcal{H}$ be a commutative Hopf algebra. If $\mathcal{H}$ is semisimple then $\mathcal{H}$ has enough cocommutative elements and $\mathcal{R}_\mathcal{H}$ is a semisimple $\mathbb{K}$-algebra.*

Proof. Let us assume first that $\mathcal{H}$ is semisimple. Then $\mathcal{H}$ is finite-dimensional by [Ş1, Remark 3.8(b)]. Let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$ and let $\overline{\mathcal{H}} := \overline{\mathbb{K}} \otimes_\mathbb{K} \mathcal{H}$. Hence $\overline{\mathcal{H}}$ is a finite-dimensional commutative Hopf algebra over $\overline{\mathbb{K}}$. Since $\mathcal{H}$ is semisimple it follows that $\overline{\mathcal{H}}$ is semisimple too. Thus, the dual Hopf algebra $\overline{\mathcal{H}}^*$ is cosemisimple and cocommutative. Since $\overline{\mathbb{K}}$ is algebraically closed, there is a finite group G such that $\overline{\mathcal{H}} = (\overline{\mathbb{K}}G)^*$. By the previous lemma, $\mathcal{R}_{\overline{\mathcal{H}}}^+ \overline{\mathcal{H}} = \overline{\mathcal{H}}^+$ and $\mathcal{R}_{\overline{\mathcal{H}}}$ is semisimple. As

$$\overline{\mathbb{K}} \otimes_\mathbb{K} (\mathcal{H}^+ / \mathcal{R}_\mathcal{H}^+ \mathcal{H}) \cong (\overline{\mathbb{K}} \otimes_\mathbb{K} \mathcal{H}^+) / (\overline{\mathbb{K}} \otimes_\mathbb{K} \mathcal{R}_\mathcal{H}^+ \mathcal{H}) \cong \overline{\mathcal{H}}^+ / \mathcal{R}_{\overline{\mathcal{H}}}^+ \overline{\mathcal{H}} = 0$$

we get $\mathcal{R}_{\mathcal{H}}^{+}\mathcal{H} = \mathcal{H}^{+}$. To prove that $\mathcal{R}_{\mathcal{H}}$ is semisimple, we can proceed as in the proof of Proposition 2.24.

If $\mathcal{H}$ is finite-dimensional over a field of characteristic zero then it is semisimple (and cosemisimple). Hence we can apply the first part of the proposition. $\quad\square$

REMARK 2.32. For a commutative Hopf algebra $\mathcal{H}$ over a field $\mathbb{K}$, we have $S_{\mathcal{H}}^{2} = \mathrm{Id}_{\mathcal{H}}$. If $\mathcal{H}$ is finite-dimensional then the trace of $S_{\mathcal{H}}^{2}$ equals $(\dim \mathcal{H})1_{\mathbb{K}}$. Therefore, by [DNR, Theorem 7.4.1], $\mathcal{H}$ is semisimple and cosemisimple if and only if $\dim \mathcal{H}$ is not zero in $\mathbb{K}$. In this case, $\mathcal{H}$ has enough cocommutative elements.

THEOREM 2.33. *Let $\mathcal{B} \subseteq \mathcal{A}$ be an $\mathcal{H}$-Galois extension, where $\mathcal{H}$ is a commutative Hopf algebra of finite dimension over a field $\mathbb{K}$ such that $\dim \mathcal{H}$ is not zero in $\mathbb{K}$. If V is a left $\mathcal{H}$-comodule and M is a Hopf bimodule then there is an isomorphisms of $\mathcal{Z}_0$-modules*

$$\mathbb{K} \otimes_{\mathcal{R}_{\mathcal{H}}} \mathrm{HH}_n(\mathcal{B}, M\square_{\mathcal{H}}V) \cong \mathrm{HH}_n(\mathcal{A}, M)\square_{\mathcal{H}}V. \tag{31}$$

Proof. In view of the above remark, $\mathcal{H}$ has enough cocommutative elements and $\mathcal{H}$ is semisimple and cosemisimple. Thus V is injective and $\mathcal{B} \subseteq \mathcal{A}$ is a faithfully flat extension. Since $\mathcal{R}_{\mathcal{H}}$ is semisimple, $\mathrm{Tor}_p^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, \mathcal{H}) = 0$ for $p > 0$, so we can apply Theorem 2.23. Furthermore, for $p > 0$

$$\mathrm{Tor}_p^{\mathcal{R}_{\mathcal{H}}}(\mathbb{K}, \mathrm{HH}_q(\mathcal{B}, M\square_{\mathcal{H}}V)) = 0,$$

as any $\mathcal{R}_{\mathcal{H}}$-module is projective. It follows that the spectral sequence in Theorem 2.23 collapses, its edge maps giving the isomorphism in (31). Obviously these maps are $\mathcal{Z}_0$-linear, as the spectral sequence lives in $_{\mathcal{Z}_0}\mathfrak{M}$ by construction. $\quad\square$

2.34. Recall that if G is a finite group of algebra automorphisms of $\mathcal{A}$ and $\mathcal{B} = \mathcal{A}^G$ then $\mathcal{A}$ is a $(\mathbb{K}G)^*$-comodule algebra and $\mathcal{B} := \mathcal{A}^{\mathrm{co}(\mathbb{K}G)^*}$. Note that the corresponding coaction $\rho : \mathcal{A} \longrightarrow \mathcal{A} \otimes (\mathbb{K}G)^*$ satisfies the relation

$$\rho(a) = \sum_{x \in G} x(a) \otimes p_x,$$

where $\{p_x \mid x \in G\}$ is the dual basis of $\{x \mid x \in G\} \subseteq \mathbb{K}G$. It is not difficult to see that $\mathcal{B} \subseteq \mathcal{A}$ is $(\mathbb{K}G)^*$-Galois if and only if there are elements $a'_1, \ldots, a'_n$ and $a''_1, \ldots, a''_n$ in $\mathcal{A}$ such that

$$\sum_{i=1}^n a'_i g(a''_i) = \delta_{g,1},$$

for all $g \in G$. Thus $(\mathbb{K}G)^*$-Galois extensions generalize Galois extensions of commutative rings. For the definition of Galois extension of commutative rings, the reader is referred to [DeMI, Chapter III]. More particularly, a finite field extension is $(\mathbb{K}G)^*$-Galois if and only if it is separable and normal. In this case, the Galois group of the extension is G, cf. [DNR, Example 6.4.3 (1)]. For this reason, in this paper, $(\mathbb{K}G)^*$-Galois extensions will be called (classical) G-Galois extensions.

Note that an object in $_{\mathcal{A}}\mathfrak{M}_{\mathcal{A}}^{(\mathbb{K}G)^*}$ is an $\mathcal{A}$-bimodule M together with a G-action on M such that, for $g \in G$, $a \in \mathcal{A}$ and $m \in M$

$$g \cdot (am) = g(a)\,[g \cdot m] \qquad \text{and} \qquad g \cdot (ma) = [g \cdot m]\,g(a).$$

We shall say that such an M is a $(G, \mathcal{A})$-bimodule.

COROLLARY 2.35. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a G-Galois extension over a field $\mathbb{K}$ such that the order of G is not zero in $\mathbb{K}$. If M is a $(G, \mathcal{A})$-bimodule then*

$$\mathrm{HH}_n(\mathcal{A}, M)^G \cong p_1 \cdot \mathrm{HH}_n(\mathcal{B}, M^G)$$

as $\mathcal{Z}_0$-modules, where $\{p_x \mid x \in G\}$ is the dual basis of the canonical basis on $\mathbb{K}G$.

Proof. By Theorem 2.33,

$$\mathbb{K} \otimes_{\mathcal{R}_{(\mathbb{K}G)^*}} \mathrm{HH}_n(\mathcal{B}, M\square_{(\mathbb{K}G)^*}\mathbb{K}) \cong \mathrm{HH}_n(\mathcal{A}, M)\square_{(\mathbb{K}G)^*}\mathbb{K}.$$

On the other hand, by the proof of Lemma 2.30, we get $\mathcal{R}_{(\mathbb{K}G)^*} \cong \mathbb{K}^{\#T(G)}$, as

$$\mathcal{S} := \{p_\sigma \mid \sigma \in T(G)\}$$

is a basis on $(\mathbb{K}G)^*$ and a complete set of orthogonal idempotents. Therefore, for a $(\mathbb{K}G)^*$-module W, we have $W = \bigoplus_{\sigma \in T(G)} p_\sigma \cdot W$. Clearly,

$$\mathbb{K} \otimes_{\mathcal{R}_{(\mathbb{K}G)^*}} W \cong \left(\mathcal{R}_{(\mathbb{K}G)^*}/\mathcal{R}^+_{(\mathbb{K}G)^*}\right) \otimes_{\mathcal{R}_{(\mathbb{K}G)^*}} W \cong W/\left(\mathcal{R}^+_{(\mathbb{K}G)^*}W\right) \cong p_1 \cdot W.$$

Note that for the last isomorphism we used that $\mathcal{R}^+_{(\mathbb{K}G)^*}$ is spanned by $\mathcal{S} \setminus \{p_1\}$.

We conclude the proof in view of the foregoing remarks and of the isomorphisms

$$X\square_{(\mathbb{K}G)^*}V \cong X^{\mathrm{co}(\mathbb{K}G)^*} \cong X^G.$$

In the above identifications, for a right $(\mathbb{K}G)^*$-comodule X, the G-invariants are taken with respect to the left G-action on X that corresponds to the $(\mathbb{K}G)^*$-comodule structure on X via the isomorphism of categories $\mathfrak{M}^{(\mathbb{K}G)^*} \cong {}_{\mathbb{K}G}\mathfrak{M}$. $\square$

3. CENTRALLY HOPF-GALOIS EXTENSIONS

Throughout this section we fix a commutative Hopf algebra $\mathcal{H}$. In the case when $\mathcal{H}$ is a finite-dimensional Hopf algebra and $\mathcal{B} \subseteq \mathcal{A}$ is an $\mathcal{H}$-comodule algebra we shall prove that $\mathcal{Z}$, the center of $\mathcal{A}$, is an $\mathcal{H}$-subcomodule. For a given Hopf bimodule M, our main purpose is to show that, under some assumptions on $\mathcal{H}$ and $\mathcal{Z}^{\mathrm{co}\mathcal{H}} \subseteq \mathcal{Z}$, the homology groups $\mathrm{HH}_*(\mathcal{A}, M)^{\mathrm{co}\mathcal{H}}$ and $\mathrm{HH}_*(\mathcal{B}, M^{\mathrm{co}\mathcal{H}})$ are isomorphic. A similar result will be proved for cyclic homology.

PROPOSITION 3.1. *Let $\mathcal{B} \subseteq \mathcal{A}$ be an $\mathcal{H}$-comodule algebra. Let $\mathcal{Z}$ denote the center of $\mathcal{A}$ and set $\mathcal{Z}' := \mathcal{Z} \bigcap \mathcal{B}$.*

(1) *If $\mathcal{H}$ is commutative and finitely generated as an algebra then $\mathcal{Z}$ is an $\mathcal{H}$-subcomodule of $\mathcal{A}$.*

(2) *If $\mathcal{Z}$ is an $\mathcal{H}$-subcomodule of $\mathcal{A}$ and $\mathcal{Z}' \subseteq \mathcal{Z}$ is an $\mathcal{H}$-Galois extension then $\mathcal{H}$ is commutative. Let us assume, in addition, that $\mathcal{Z}' \subseteq \mathcal{Z}$ is a faithfully flat extension. Then $\mathcal{A}^{\mathrm{co}\mathcal{H}} \subseteq \mathcal{A}$ is a faithfully flat $\mathcal{H}$-Galois extension.*

Proof. (1) As $\mathcal{A}$ is an $\mathcal{H}$-comodule, $(\mathcal{A}, \cdot)$ is a left $\mathcal{H}^*$-module, where for α in $\mathcal{H}^*$ and a in $\mathcal{A}$

$$\alpha \cdot a = \sum \alpha(a_{\langle 1 \rangle})a_{\langle 0 \rangle}. \tag{32}$$

To prove that $\mathcal{Z}$ is an $\mathcal{H}$-subcomodule we must check that $\mathcal{Z}$ is an $\mathcal{H}^*$-submodule. Let $\mathcal{H}^\circ$ denote the finite dual of $\mathcal{H}$ (for the definition of the finite dual of an algebra see [DNR, Section 1.5]). It is well-known that $\mathcal{H}^\circ$ is an $S_\mathcal{H}$-invariant subalgebra of

$\mathcal{H}^*$, so it has a canonical structure of Hopf algebra. The comultiplication of $\mathcal{H}^\circ$ is uniquely defined such that $\Delta(\alpha) := \sum_{i=1}^{n} \alpha_i' \otimes \alpha_i''$ if and only if

$$\alpha(xy) = \sum_{i=1}^{n} \alpha_i'(x)\alpha_i''(y),$$

for all $x, y \in \mathcal{H}$. Clearly, $\mathcal{A}$ is an $\mathcal{H}^\circ$-module. In fact $\mathcal{A}$ is an $\mathcal{H}^\circ$-module algebra, that is

$$\alpha \cdot (a'a'') = \sum (\alpha_{(1)} \cdot a')(\alpha_{(2)} \cdot a''), \tag{33}$$

for $\alpha \in \mathcal{H}^\circ$ and $a', a'' \in \mathcal{A}$. We now want to show that $\mathcal{Z}$ is an $\mathcal{H}^\circ$-submodule. For $\alpha \in \mathcal{H}^\circ$ and $a \in \mathcal{Z}$, we get

$$\begin{aligned}(\alpha \cdot a)x &= \sum (\alpha_{(1)} \cdot a)[\alpha_{(2)} \cdot (S_{\mathcal{H}^\circ}\alpha_{(3)} \cdot x)] \\ &= \sum \alpha_{(1)} \cdot [a(S_{\mathcal{H}^\circ}\alpha_{(2)} \cdot x)] \\ &= \sum \alpha_{(1)} \cdot [(S_{\mathcal{H}^\circ}\alpha_{(2)} \cdot x)a],\end{aligned}$$

where for the second equality we used (33). By [Ab, Corollary 2.3.17(ii)], $\mathcal{H}^\circ$ is cocommutative. Thus

$$\begin{aligned}\sum \alpha_{(1)} \cdot [(S_{\mathcal{H}^\circ}\alpha_{(2)} \cdot x)a] &= \sum [(\alpha_{(1)}S_{\mathcal{H}^\circ}\alpha_{(3)}) \cdot x](\alpha_{(2)} \cdot a) \\ &= \sum [(\alpha_{(1)}S_{\mathcal{H}^\circ}\alpha_{(2)}) \cdot x](\alpha_{(3)} \cdot a) \\ &= x(\alpha \cdot a).\end{aligned}$$

By the foregoing computation, we conclude that $\alpha \cdot a \in \mathcal{Z}$. Since $\mathcal{H}$ is finitely generated as an algebra it follows that $\mathcal{H}^\circ$ is dense in $\mathcal{H}^*$, with respect to the finite topology, cf. [Ab, Theorems 2.2.17 and 2.3.19]. This means that, for every $\alpha \in \mathcal{H}^*$ and every finite set $X \subseteq \mathcal{H}$ there is $\beta \in \mathcal{H}^\circ$ such that $\alpha = \beta$ on X. We can now prove that $\mathcal{Z}$ is an $\mathcal{H}^*$-submodule of $\mathcal{A}$. Let $\alpha \in \mathcal{H}^*$ and $a \in \mathcal{A}$. If $\rho(a) = \sum_{i=1}^{n} a_i \otimes h_i$, then there is $\beta \in \mathcal{H}^\circ$ such that $\alpha(h_i) = \beta(h_i)$, for every $i = 1, \ldots, n$. Thus

$$\alpha \cdot a = \sum_{i=1}^{n} \alpha(h_i)a_i = \sum_{i=1}^{n} \beta(h_i)a_i = \beta \cdot a.$$

It follows that $\alpha \cdot a \in \mathcal{Z}$, as $\beta \in \mathcal{H}^\circ$ and $a \in \mathcal{Z}$.

(2) Since $\mathcal{Z}$ is a subcomodule of $\mathcal{A}$, it follows that $\mathcal{Z}^{\mathrm{co}\mathcal{H}} = \mathcal{Z}'$. The canonical map $\beta_{\mathcal{Z}} : \mathcal{Z} \otimes_{\mathcal{Z}'} \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{H}$, that corresponds to the $\mathcal{H}$-comodule algebra $\mathcal{Z}' \subseteq \mathcal{Z}$, is bijective by assumption. As $\mathcal{Z}$ is a commutative algebra, $\beta_{\mathcal{Z}}$ is a morphism of algebras and $\mathcal{Z} \otimes_{\mathcal{Z}'} \mathcal{Z}$ is commutative. We conclude that $\mathcal{H}$ is commutative by remarking that $\mathcal{H}$ is a subalgebra of $\mathcal{Z} \otimes \mathcal{H}$, which is commutative.

We now assume that $\mathcal{Z}' \subseteq \mathcal{Z}$ is a faithfully flat $\mathcal{H}$-Galois extension. For each $h \in \mathcal{H}$ there are $a_1', \ldots, a_r'$ and $a_1'', \ldots, a_r''$ in $\mathcal{Z}$ such that

$$\beta_{\mathcal{Z}}(\sum_{i=1}^{r} a_i' \otimes_{\mathcal{Z}^{\mathrm{co}\mathcal{H}}} a_i'') = 1 \otimes h. \tag{34}$$

Obviously, $\beta_{\mathcal{A}}(\sum_{i=1}^{r} a_i' \otimes_{\mathcal{B}} a_i'') = \beta_{\mathcal{Z}}(\sum_{i=1}^{r} a_i' \otimes_{\mathcal{Z}^{\mathrm{co}\mathcal{H}}} a_i'') = 1 \otimes h$ and $\beta_{\mathcal{A}}$ is a morphism of left $\mathcal{A}$-modules. Thus $\beta_{\mathcal{A}}$ is surjective too. Since $\mathcal{Z}' \subseteq \mathcal{Z}$ is faithfully flat it follows that $\mathcal{Z}$ is injective as an $\mathcal{H}$-comodule. By [SS, Lemma 4.1.] there is an $\mathcal{H}$-comodule map $\phi : \mathcal{H} \to \mathcal{Z}$ such that $\phi(1) = 1$. We may regard ϕ as an $\mathcal{H}$-colinear map from $\mathcal{H}$ to $\mathcal{A}$, so $\mathcal{A}$ is injective as an $\mathcal{H}$-comodule. Hence $\mathcal{B} \subseteq \mathcal{A}$ is a faithfully flat $\mathcal{H}$-Galois extension, cf. [SS, Theorem 4.10]. $\square$

DEFINITION 3.2. Let $\mathcal{H}$ be a commutative Hopf algebra. We say that an $\mathcal{H}$-comodule algebra $\mathcal{B} \subseteq \mathcal{A}$ is a *centrally $\mathcal{H}$-Galois extension* if the center $\mathcal{Z}$ of $\mathcal{A}$ is a subcomodule and $\mathcal{Z}' \subseteq \mathcal{Z}$ is a faithfully flat $\mathcal{H}$-Galois extension, where $\mathcal{Z}' := \mathcal{Z}^{\mathrm{co}\mathcal{H}}$.

REMARK 3.3. In the case when $\mathcal{H}$ is cosemisimple and finitely generated as an algebra, an $\mathcal{H}$-comodule algebra $\mathcal{A}$ is centrally $\mathcal{H}$-Galois if and only if $\mathcal{Z}' \subseteq \mathcal{Z}$ is $\mathcal{H}$-Galois.

3.4. Throughout the remaining part of this section we fix a commutative Hopf algebra $\mathcal{H}$ and a centrally $\mathcal{H}$-Galois extension $\mathcal{B} \subseteq \mathcal{A}$. We also fix a Hopf bimodule M and a left $\mathcal{H}$-comodule V.

We have seen that $\mathcal{B} \subseteq \mathcal{A}$ is $\mathcal{H}$-Galois, so $\mathrm{HH}_n(\mathcal{B}, M)$ is a left $\mathcal{H}$-module. Our aim now is to give an equivalent description of this action. We fix $h \in \mathcal{H}$ and we pick up $a_1', \ldots, a_r'$ and $a_1'', \ldots, a_r''$ in $\mathcal{Z}$ such that (34) holds true. We now define $\lambda_n^h(M) : C_n(\mathcal{B}, M) \to C_n(\mathcal{B}, M)$ by

$$\lambda_n^h(M)(m \otimes b^1 \otimes \cdots \otimes b^n) = \sum_{i=1}^r a_i'' m a_i' \otimes b^1 \otimes \cdots \otimes b^n.$$

It is easy to see that $\lambda_*^h(M)$ is a morphism of complexes, as a_i' and a_i'' are in the center of $\mathcal{A}$ for all $i = 1, \ldots, r$. Let $\bar{\lambda}_n^h(M)$ be the endomorphism of $\mathrm{HH}_n(\mathcal{B}, M)$ induced by $\lambda_n^h(M)$. Clearly, both λ_n^h and $\bar{\lambda}_n^h$ are natural transformations.

PROPOSITION 3.5. *Let $h \in \mathcal{H}$. For a Hopf bimodule M and $\omega \in \mathrm{HH}_n(\mathcal{B}, M)$*

$$h \cdot \omega = \bar{\lambda}_n^h(M)(\omega).$$

If in addition M is a symmetric $\mathcal{Z}$-bimodule then the above action is trivial. In this case, for an injective left $\mathcal{H}$-comodule V, the action of $\mathcal{R}_\mathcal{H}$ on $\mathrm{HH}_n(\mathcal{B}, M \square_\mathcal{H} V)$ is trivial too.

Proof. Let μ_*^h be the natural transformations that lift the $\mathcal{H}$-action on $M_\mathcal{B}$, as in the proof of Proposition 2.12. Thus, for ω in $\mathrm{HH}_n(\mathcal{B}, M)$

$$\mu_n^h(M)(\omega) = h \cdot \omega.$$

We shall prove by induction on n that $\bar{\lambda}_n^h(M) = \mu_n^h(M)$. For $n = 0$ that is obvious, by construction of $\bar{\lambda}_0^h$ and the definition of the $\mathcal{H}$-module structure in (12). Let us assume that $\bar{\lambda}_n^h(K) = \mu_n^h(K)$, for any Hopf bimodule K. Since $\mathcal{B} \subseteq \mathcal{A}$ is a faithfully flat $\mathcal{H}$-Galois extension, $U := \mathcal{A} \otimes \mathcal{A}$ is a projective generator in $_\mathcal{A}\mathcal{M}_\mathcal{A}^\mathcal{H}$. Thus, there is an exact sequence

$$0 \longrightarrow K_0 \longrightarrow L \longrightarrow M \longrightarrow 0$$

in $_\mathcal{A}\mathcal{M}_\mathcal{A}^\mathcal{H}$ such that $L \cong U^{(I)}$, where I is a certain set. On the other hand, $\mathcal{A}$ is projective as a left and right $\mathcal{B}$-module, so U is projective as a $\mathcal{B}$-bimodule. Hence $\mathrm{HH}_n(\mathcal{B}, L) = 0$, for $n > 0$. Consequently, $\delta_{n+1} : \mathrm{HH}_{n+1}(\mathcal{B}, M) \to \mathrm{HH}_n(\mathcal{B}, K_0)$ is injective. On the other hand, by construction, μ_*^h is a morphism of δ-functors. Thus

$$\delta_{n+1} \circ \mu_{n+1}^h(M) = \mu_n^h(K_0) \circ \delta_{n+1}. \tag{35}$$

Since the long exact sequence in homology is natural and λ_*^h is a natural morphism of complexes

$$\delta_{n+1} \circ \bar{\lambda}_{n+1}^h(M) = \bar{\lambda}_n^h(K_0) \circ \delta_{n+1}. \tag{36}$$

Using relations (35) and (36), the induction hypothesis and the fact that δ_{n+1} is injective one gets $\mu_{n+1}^h(M) = \bar{\lambda}_{n+1}^h(M)$.

Let us assume that M is symmetric as a $\mathcal{Z}$-bimodule, i.e. $z \cdot m = m \cdot z$, for any $z \in \mathcal{Z}$ and $m \in M$. Thus

$$\lambda_n^h(M)(m \otimes b^1 \otimes \cdots \otimes b^n) = \sum_{i=1}^k ma_i'a_i'' \otimes b^1 \otimes \cdots \otimes b^n = \varepsilon(h)m \otimes b^1 \otimes \cdots \otimes b^n,$$

where for the second equality we used [JȘ, Relation (5)]. Thus $\bar{\lambda}_n^h(M)(\omega) = \varepsilon(h)\omega$. By the first part of the proposition we deduce that the action of $\mathcal{H}$ on $\mathrm{HH}_*(\mathcal{B}, M)$ is trivial. Finally, if V is an injective left $\mathcal{H}$-comodule, then there is an isomorphism

$$\mathrm{HH}_*(\mathcal{B}, M\square_{\mathcal{H}}V) \cong \mathrm{HH}_*(\mathcal{B}, M)\square_{\mathcal{H}}V$$

of $\mathcal{R}_{\mathcal{H}}$-modules. Note that $\mathrm{HH}_*(\mathcal{B}, M)\square_{\mathcal{H}}V$ is a $\mathcal{R}_{\mathcal{H}}$-submodule of $\mathrm{HH}_*(\mathcal{B}, M) \otimes V$. Hence, the action of $\mathcal{R}_{\mathcal{H}}$ on $\mathrm{HH}_*(\mathcal{B}, M)\square_{\mathcal{H}}V$ is also trivial. $\square$

THEOREM 3.6. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a centrally $\mathcal{H}$-Galois extension, where $\mathcal{H}$ is a finite-dimensional Hopf algebra over a field $\mathbb{K}$ such that $\dim \mathcal{H}$ is not zero in $\mathbb{K}$. Let M be a Hopf bimodule which is symmetric as a $\mathcal{Z}$-bimodule. If V is a left $\mathcal{H}$-comodule then there are isomorphisms of $\mathcal{Z}'$-modules*

$$\mathrm{HH}_*(\mathcal{A}, M)\square_{\mathcal{H}}V \simeq \mathrm{HH}_*(\mathcal{B}, M\square_{\mathcal{H}}V). \tag{37}$$

Proof. We have already noticed that $\mathcal{B} \subseteq \mathcal{A}$ is a faithfully flat $\mathcal{H}$-Galois extension and that $\mathrm{H}_*(\mathcal{B}, M\square_{\mathcal{H}}V)$ is a trivial $\mathcal{R}_{\mathcal{H}}$-module. The isomorphism of left $\mathcal{Z}'$-modules(37) follows by applying Theorem 2.33. $\square$

COROLLARY 3.7. *Keeping the notation and the assumptions from the preceding theorem, there are isomorphisms of $\mathcal{Z}'$-modules*

$$\mathrm{HH}_*(\mathcal{A}, M)^{\mathrm{co}\mathcal{H}} \simeq \mathrm{HH}_*(\mathcal{B}, M^{\mathrm{co}\mathcal{H}}).$$

Proof. Take $V = \mathbb{K}$ in Theorem 3.6 and note that $(-)^{\mathrm{co}\mathcal{H}} \cong (-)\square_{\mathcal{H}}\mathbb{K}$. $\square$

REMARK 3.8. A faithfully flat $\mathcal{H}$-Galois extension of commutative algebras is centrally Hopf-Galois. Thus the isomorphisms in the preceding corollary exist for such an extension, provided that $\mathcal{H}$ is finite-dimensional and $\dim \mathcal{H}$ is not zero in $\mathbb{K}$.

COROLLARY 3.9. *Let G be a finite group of automorphisms of an algebra $\mathcal{A}$ over a field $\mathbb{K}$ such that the order of G is not zero in $\mathbb{K}$. Let $\mathcal{Z}$ denote the center of $\mathcal{A}$ and let M be an $(G, \mathcal{A})$-Hopf bimodule. If $\mathcal{Z}^G \subseteq \mathcal{Z}$ is a G-Galois extension then there are isomorphisms of $\mathcal{Z}^G$-modules*

$$\mathrm{HH}_*(\mathcal{A}, M)^G \simeq \mathrm{HH}_*(\mathcal{A}^G, M^G).$$

Proof. Apply Corollary 3.7 for $\mathcal{H} := (\mathbb{K}G)^*$. Note that, for a $(\mathbb{K}G)^*$-comodule X, we have $X^{\mathrm{co}(\mathbb{K}G)^*} = X^G$, cf. §2.34. $\square$

REMARK 3.10. Note that the proof of the previous corollary works only if the order of G is not invertible in $\mathbb{K}$, as $\mathbb{K}$ must be injective as $(\mathbb{K}G)^*$-comodule in order to apply Theorem 3.6. On the other hand the isomorphisms in [Lo2, §6] hold true without any assumption on the characteristic of $\mathbb{K}$.

THEOREM 3.11. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a centrally Galois extension over a finite-dimensional Hopf algebra $\mathcal{H}$ such that $\dim \mathcal{H}$ is not zero in $\mathbb{K}$. Let M be a Hopf bimodule which is symmetric as a $\mathcal{Z}$-bimodule. Then, there are isomorphisms of $\mathcal{Z}$-modules and $\mathcal{H}$-comodules*

$$\mathrm{HH}_*(\mathcal{A}, M) \simeq \mathcal{Z} \otimes_{\mathcal{Z}'} \mathrm{HH}_*(\mathcal{B}, M^{\mathrm{co}\mathcal{H}}).$$

Proof. Since $\mathcal{Z}' \subseteq \mathcal{Z}$ is a faithfully flat $\mathcal{H}$-Galois extension the categories $_{\mathcal{Z}}\mathfrak{M}^{\mathcal{H}}$ and $_{\mathcal{Z}'}\mathfrak{M}$ are equivalent, cf. [SS, Theorems 4.9 and 4.10]. More precisely,

$$\mathcal{Z} \otimes_{\mathcal{Z}'} (-) : {}_{\mathcal{Z}'}\mathfrak{M} \longrightarrow {}_{\mathcal{Z}}\mathfrak{M}^{\mathcal{H}}$$

is an equivalence of categories, whose inverse is the functor $X \mapsto X^{\mathrm{co}H}$. Thus,

$$\mathrm{H}_*(\mathcal{A}, M) \cong \mathcal{Z} \otimes_{\mathcal{Z}'} \mathrm{H}_*(\mathcal{A}, M)^{\mathrm{co}\mathcal{H}}.$$

We conclude de proof in view of Corollary 3.7. $\qquad\square$

3.12. Recall that $\mathcal{B} \subseteq \mathcal{A}$ is a centrally $\mathcal{H}$-Galois extension. In particular, $\mathcal{H}$ is commutative. Therefore, the map $t_n : C_n(\mathcal{A}, \mathcal{A}) \longrightarrow C_n(\mathcal{A}, \mathcal{A})$ given by

$$t_n(a^0 \otimes a^1 \otimes \cdots \otimes a^n) = a^n \otimes a^0 \otimes \cdots \otimes a^{n-1}.$$

is a morphism of right $\mathcal{H}$-comodules, where $\mathcal{H}$ coacts on $\mathcal{A}^{\otimes n+1}$ as in (2.15). Cyclic homology of $\mathcal{A}$, denoted by $\mathrm{HC}_*(\mathcal{A})$, is defined as the homology of the total complex of the bicomplex $\mathrm{CC}_{**}(\mathcal{A})$, see [We, Definition 9.6.6]. As the operator t_n is $\mathcal{H}$-colinear for every n, it follows that $\mathrm{CC}_{**}(\mathcal{A})$ is a bicomplex in the category of right $\mathcal{H}$-comodules. Thus $\mathrm{HC}_n(\mathcal{A})$ is an $\mathcal{H}$-comodule too.

We can now prove the following.

THEOREM 3.13. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a centrally $\mathcal{H}$-Galois extension, where $\mathcal{H}$ is a finite-dimensional Hopf algebra such that $\dim \mathcal{H}$ is not zero in $\mathbb{K}$. Then*

$$\mathrm{HC}_n(\mathcal{A})^{\mathrm{co}\mathcal{H}} \cong \mathrm{HC}_n(\mathcal{B}). \tag{38}$$

Proof. The case $n = 0$ is obvious, in view of Corollary 3.7 and of the fact that cyclic homology and Hochschild homology are equal in degree zero. For each right $\mathcal{H}$-comodule X the natural transformation

$$\nu(X) : X^{\mathrm{co}\mathcal{H}} \longrightarrow X \square_{\mathcal{H}} \mathbb{K}, \qquad \nu(X)(x) := x \otimes 1$$

is an isomorphism. Since $\mathcal{H}$ is commutative and the characteristic of $\mathbb{K}$ does not divide the dimension of $\mathcal{H}$, we deduce that $\mathcal{H}$ is cosemisimple. Hence $\mathbb{K}$ is an injective comodule. Thus the functor that maps a right $\mathcal{H}$-comodule X to $X \square_{\mathcal{H}} \mathbb{K}$ is exact. Consequently, by applying the functor $(-)^{\mathrm{co}\mathcal{H}}$ to Connes' exact sequence [We, Proposition 9.6.11], we get the exact sequence on the top of the following diagram.

$$
\begin{array}{ccccccccc}
\widetilde{\mathrm{HC}}_n(\mathcal{A}) & \xrightarrow{\tilde{B}} & \widetilde{\mathrm{HH}}_{n+1}(\mathcal{A}) & \xrightarrow{\tilde{I}} & \widetilde{\mathrm{HC}}_{n+1}(\mathcal{A}) & \xrightarrow{\tilde{S}} & \widetilde{\mathrm{HC}}_{n-1}(\mathcal{A}) & \xrightarrow{\tilde{B}} & \widetilde{\mathrm{HH}}_n(\mathcal{A}) \\
{\cong}\big\uparrow & & {\cong}\big\uparrow & & \big\uparrow & & {\cong}\big\uparrow & & {\cong}\big\uparrow \\
\mathrm{HC}_n(\mathcal{B}) & \xrightarrow{B} & \mathrm{HH}_{n+1}(\mathcal{B}) & \xrightarrow{I} & \mathrm{HC}_{n+1}(\mathcal{B}) & \xrightarrow{S} & \mathrm{HC}_{n-1}(\mathcal{B}) & \xrightarrow{B} & \mathrm{HH}_n(\mathcal{B})
\end{array}
$$

Here, $\widetilde{\mathrm{HC}}_*(\mathcal{A})$ and $\widetilde{\mathrm{HH}}_*(\mathcal{A})$ denote $\mathrm{HC}_*(\mathcal{A})^{\mathrm{co}\mathcal{H}}$ and $\mathrm{HH}_*(\mathcal{A})^{\mathrm{co}\mathcal{H}}$, respectively. Note that by induction hypothesis, the first and the fourth vertical arrows are isomorphisms. Furthermore, by taking $M = \mathcal{A}$ in Corollary 3.7, we get that the second

and the fifth vertical maps are isomorphisms. Thus, by 5-Lemma [We, p.13] the vertical map in the middle is also an isomorphism. $\qquad\square$

We conclude this paper showing that, under some extra assumptions, Ore extensions provide non-trivial examples of centrally Hopf-Galois extensions. To define an Ore extension of a $\mathbb{K}$-algebra $\mathcal{A}$, we need an algebra automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ and a σ-derivation $\delta : \mathcal{A} \to \mathcal{A}$. Recall that δ is a σ-derivation if, for a and b in $\mathcal{A}$,

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b.$$

For σ and δ as above one defines a new algebra $\mathcal{A}[X, \sigma, \delta]$, the Ore extension of $\mathcal{A}$. As a left $\mathcal{A}$-module, $\mathcal{A}[X, \sigma, \delta]$ is free with basis $\{1, X, X^2, \dots\}$ and its multiplication is the unique left $\mathcal{A}$-linear morphism such that $X^n X^m = X^{n+m}$ and

$$Xa = \sigma(a)X + \delta(a). \tag{39}$$

For simplicity, we shall denote the Ore extension $\mathcal{A}[X, \sigma, \delta]$ by $\mathcal{T}$.

We now assume, in addition, that $\mathcal{A}$ is an $\mathcal{H}$-comodule algebra and that σ and δ are morphisms of comodules. Set $\mathcal{B} := \mathcal{A}^{\mathrm{co}\mathcal{H}}$. Since σ and δ are morphisms of $\mathcal{H}$-comodules they map $\mathcal{B}$ into $\mathcal{B}$. We still denote the restrictions of these maps to $\mathcal{B}$ by σ and δ. Clearly, δ can be regarded as a σ-derivation of $\mathcal{B}$, so we can construct the Ore extension $\mathcal{S} := \mathcal{B}[X, \sigma, \delta]$.

LEMMA 3.14. *The comodule structure map* $\rho_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{H}$ *can be extended in a unique way to an $\mathcal{H}$-coaction $\rho_{\mathcal{T}}$ on $\mathcal{T}$ such that, for $a \in \mathcal{A}$ and $n \in \mathbb{N}$,*

$$\rho_{\mathcal{T}}(aX^n) = \sum a_{\langle 0 \rangle} X^n \otimes a_{\langle 1 \rangle}. \tag{40}$$

With respect to this coaction the subalgebra of coinvariant elements in $\mathcal{T}$ is $\mathcal{S}$.

Proof. For $n \in \mathbb{N}$ and $0 \le k \le n$ let $f_k^{(n)}$ be the non-commutative polynomial in σ and δ with coefficients in the prime subfield of $\mathbb{K}$ such that

$$X^n a = \sum_{k=0}^n f_k^{(n)}(a) X^k. \tag{41}$$

Let us put $f_{-1}^{(n)} = f_{n+1}^{(n)} = 0$. Thus, by multiplying to the left both sides of (41) by X and using (39), for $0 \le k \le n + 1$, we get

$$f_k^{(n+1)} = \sigma f_{k-1}^{(n)} + \delta f_k^{(n)},$$

For $a^0, \dots, a^n$ in $\mathcal{A}$ we now define

$$\rho_{\mathcal{T}}\left(\sum_{i=0}^n a^i X^i\right) = \sum_{i=0}^n a^i_{\langle 0 \rangle} X^i \otimes a^i_{\langle 1 \rangle}.$$

Clearly, $\rho_{\mathcal{T}}$ defines a coaction of $\mathcal{H}$ on $\mathcal{T}$ and verifies the identity (40). We have to prove that $\rho_{\mathcal{T}}$ is a morphism of algebras, i.e. $\rho_{\mathcal{T}}(fg) = \rho_{\mathcal{T}}(f)\rho_{\mathcal{T}}(g)$ for any $f, g \in \mathcal{T}$. In fact, it is enough to prove this equality for $f = X^n$ and $g = a$, with $a \in \mathcal{A}$ and $n \in \mathbb{N}^*$. Since $f_k^{(n)}$ are non-commutative polynomials in σ and δ and these maps are morphism of $\mathcal{H}$-modules, it follows that $f_k^{(n)}$ are also $\mathcal{H}$-colinear. Hence,

$$\begin{aligned}
\rho_{\mathcal{T}}(X^n a) &= \rho_{\mathcal{T}}\left(\sum_{k=0}^n f_k^{(n)}(a) X^k\right) \\
&= \sum_{k=0}^n f_k^{(n)}(a)_{\langle 0 \rangle} X^k \otimes f_k(a)_{\langle 1 \rangle} \\
&= \sum_{k=0}^n f_k^{(n)}(a_{\langle 0 \rangle}) X^k \otimes a_{\langle 1 \rangle}.
\end{aligned}$$

On the other hand,

$$\rho_{\mathcal{T}}(X^n)\rho_{\mathcal{A}}(a) = \sum X^n a_{\langle 0 \rangle} \otimes a_{\langle 1 \rangle} = \sum_{k=0}^{n} f_k^{(n)}(a_{\langle 0 \rangle})X^k \otimes a_{\langle 1 \rangle} = \rho_{\mathcal{T}}(X^n a).$$

Obviously $\rho_{\mathcal{T}}$ is unital. Thus $\mathcal{T}$ is an $\mathcal{H}$-comodule algebra. It remains to prove that $\mathcal{T}^{\mathrm{co}\mathcal{H}} = \mathcal{S}$. For this, we fix a basis $\{h_j \mid j \in J\}$ on $\mathcal{H}$. We may assume that there is $j_0 \in J$ such that $h_{j_0} = 1$. Let us take $f = \sum_{i=0}^{n} a^i X^i$ in $\mathcal{T}$ and write $\rho(a^i) = \sum_{j \in J} a_j^i \otimes h_j$. Therefore,

$$\rho(f) = \sum_{i=0}^{n} \sum_{j \in J} a_j^i X^i \otimes h_j.$$

It follows that $f \in \mathcal{T}^{\mathrm{co}\mathcal{H}}$ if and only if $\sum_{i=0}^{n} a_j^i X^i = \delta_{j,j_0} \sum_{i=0}^{n} a^i X^i$. Thus, f is $\mathcal{H}$-coinvariant if and only if

$$\rho(a^i) = \sum_{j \in J} \delta_{j,j_0} a^i \otimes h_j = a^i \otimes 1,$$

for all $i = 0, \ldots, n$. We deduce that $f \in \mathcal{T}^{\mathrm{co}\mathcal{H}}$ if and only if $f \in \mathcal{S}$. $\square$

LEMMA 3.15. *Let $f = \sum_{i=0}^{n} a^i X^i$ be an element in $\mathcal{T}$ and $a^{-1} = a^{n+1} = 0$. Then f is in $\mathcal{Z}(\mathcal{T})$, the center of $\mathcal{T}$, if and only if*

$$\sum_{k=i}^{n} a^k f_i^{(k)}(a) = aa^i, \qquad \text{for } i = 0, \ldots, n \text{ and } a \in \mathcal{A}, \tag{42}$$
$$\sigma(a^i) + \delta(a^{i+1}) = a^i, \qquad \text{for } i = -1, 0, \ldots, n+1. \tag{43}$$

Proof. As an algebra, $\mathcal{T}$ is generated by $\mathcal{A}$ and X. Hence, f is central if and only if $Xf = fX$ and $af = fa$, for all $a \in \mathcal{A}$. We get

$$fa = \sum_{k=0}^{n} a^k X^k a = \sum_{k=0}^{n} \sum_{i=0}^{k} a^k f_i^{(k)}(a)X^i = \sum_{i=0}^{n} \left(\sum_{k=i}^{n} a^k f_i^{(k)}(a) \right) X^i.$$

We deduce that $fa = af$ and (42) are equivalent. On the other hand, $fX = Xf$ is equivalent to

$$\sum_{i=0}^{n} \sigma(a^i)X^{i+1} + \sum_{i=0}^{n} \delta(a^i)X^i = \sum_{i=0}^{n} a^i X^{i+1}.$$

In conclusion, $fX = Xf$ and (43) are equivalent. $\square$

COROLLARY 3.16. *Let $\mathcal{A}^{\sigma} = \{a \mid \sigma(a) = a\}$ and $\mathcal{A}^{\delta} = \{a \mid \delta(a) = 0\}$. If $\mathcal{Z}$ is the center of $\mathcal{A}$ then*

$$\mathcal{Z}(\mathcal{T}) \cap \mathcal{A} = \mathcal{A}^{\sigma} \cap \mathcal{A}^{\delta} \cap \mathcal{Z}.$$

Proof. We regard $\mathcal{A}$ as a subalgebra of $\mathcal{T}$. Thus, $a^0 \in \mathcal{A}$ is in the center of $\mathcal{T}$ if and only if for any $a \in \mathcal{A}$ we have

$$\delta(a^0) = 0, \quad \sigma(a^0) = a^0, \quad aa^0 = a^0 f_0^{(0)}(a).$$

Since $f_0^{(0)} = \mathrm{Id}_A$, we get $\mathcal{Z}(\mathcal{T}) \cap \mathcal{A} = \mathcal{Z} \cap \mathcal{A}^{\sigma} \cap \mathcal{A}^{\delta}$. $\square$

THEOREM 3.17. *Let $\mathcal{B} \subseteq \mathcal{A}$ be an $\mathcal{H}$-comodule algebra, where $\mathcal{H}$ is a commutative finite-dimensional Hopf algebra over a field of characteristic zero. Let $\sigma : \mathcal{A} \to \mathcal{A}$ be an algebra map and $\delta : \mathcal{A} \to \mathcal{A}$ be a σ-derivation. Assume that both σ and δ are morphisms of $\mathcal{H}$-comodules. Let $\mathcal{T} := \mathcal{A}[X, \sigma, \delta]$ and $\mathcal{S} := \mathcal{B}[X, \sigma, \delta]$.*

(1) *The center $\mathcal{Z}$ of $\mathcal{A}$ and $\mathcal{A}^{\sigma} \cap \mathcal{A}^{\delta} \cap \mathcal{Z}$ are $\mathcal{H}$-comodule subalgebras of $\mathcal{A}$. The algebra of coinvariant elements in $\mathcal{A}^{\sigma} \cap \mathcal{A}^{\delta} \cap \mathcal{Z}$ is $\mathcal{B}^{\sigma} \cap \mathcal{B}^{\delta} \cap \mathcal{Z}$.*

(2) *If $\mathcal{B}^{\sigma} \cap \mathcal{B}^{\delta} \cap \mathcal{Z} \subseteq \mathcal{A}^{\sigma} \cap \mathcal{A}^{\delta} \cap \mathcal{Z}$ is an $\mathcal{H}$-Galois extension then the extension $\mathcal{S} \subseteq \mathcal{T}$ is a centrally $\mathcal{H}$-Galois extension.*

Proof. (1) By Proposition 3.1 (2), $\mathcal{Z}$ is an $\mathcal{H}$-subcomodule of $\mathcal{A}$ as $\mathcal{H}$ is finite-dimensional and commutative. Since σ and δ are morphisms of $\mathcal{H}$-comodules it follows that $\mathcal{A}^\sigma = \ker(\sigma - \mathrm{Id}_\mathcal{A})$ and $\mathcal{A}^\delta = \ker\delta$ are $\mathcal{H}$-subcomodules of $\mathcal{A}$. We deduce that $\mathcal{A}^\sigma \cap \mathcal{A}^\delta \cap \mathcal{Z}$ is an $\mathcal{H}$-comodule algebra. Its subalgebra of coinvariant elements is

$$[\mathcal{A}^\sigma \cap \mathcal{A}^\delta \cap \mathcal{Z}]^{\mathrm{co}\mathcal{H}} = \mathcal{A}^\sigma \cap \mathcal{A}^\delta \cap \mathcal{Z} \cap \mathcal{B} = \mathcal{B}^\sigma \cap \mathcal{B}^\delta \cap \mathcal{Z}.$$

(2) Again by Proposition 3.1 (2), the center $\mathcal{Z}(\mathcal{T})$ of $\mathcal{T}$ is an $\mathcal{H}$-subcomodule of $\mathcal{T}$. Since $\mathcal{H}$ is commutative and finite-dimensional over a field of characteristic zero, we deduce that $\mathcal{H}$ is cosemisimple. Hence, $\mathcal{Z}(\mathcal{T})$ is injective as an $\mathcal{H}$-comodule. In view of [SS, Theorem 4.10], to prove that $\mathcal{Z}(\mathcal{T}) \cap \mathcal{S} \subseteq \mathcal{Z}(\mathcal{T})$ is $\mathcal{H}$-Galois and faithfully flat, we have to show that the canonical map

$$\beta_{\mathcal{Z}(\mathcal{T})} : \mathcal{Z}(\mathcal{T}) \otimes_{\mathcal{Z}(\mathcal{T}) \cap \mathcal{S}} \mathcal{Z}(\mathcal{T}) \longrightarrow \mathcal{Z}(\mathcal{T}) \otimes \mathcal{H}$$

is surjective. Proceeding as in the proof of Proposition 3.1 (3) it is enough to show that $1 \otimes h$ is in the image of $\beta_{\mathcal{Z}(\mathcal{T})}$, for every $h \in \mathcal{H}$. Let $\mathcal{Z}' := \mathcal{A}^\sigma \cap \mathcal{A}^\delta \cap \mathcal{Z}$. By assumption, the canonical map

$$\beta_{\mathcal{Z}'} : \mathcal{Z}' \otimes_{\mathcal{Z}' \cap \mathcal{B}} \mathcal{Z}' \longrightarrow \mathcal{Z}' \otimes \mathcal{H}$$

is bijective. Thus, there are $a'_1, \ldots, a'_r$ and $a''_1, \ldots, a''_r$ in $\mathcal{Z}'$ such that

$$\beta_{\mathcal{Z}'}(\textstyle\sum_{i=1}^r a'_i \otimes_{\mathcal{Z}' \cap \mathcal{B}} a''_i) = 1 \otimes h.$$

By the previous corollary, $\mathcal{Z}'$ is an $\mathcal{H}$-submodule of $\mathcal{Z}(\mathcal{T})$. Therefore,

$$\beta_{\mathcal{Z}(\mathcal{T})}(\textstyle\sum_{i=1}^n a'_i \otimes_{\mathcal{Z}(\mathcal{T}) \cap \mathcal{S}} a''_i) = \beta_{\mathcal{Z}'}(\textstyle\sum_{i=1}^n a'_i \otimes_{\mathcal{Z}' \cap \mathcal{B}} a''_i) = 1 \otimes h.$$

Hence, the theorem is completely proven. $\square$

A more concrete example can be obtained as follows. Let $\mathbb{K} \subseteq \mathcal{K} \subseteq \mathcal{A}$ be field extensions such that $\mathcal{K} \subseteq \mathcal{A}$ is finite, separable and normal of Galois group G. We assume that $G = NH$, where H and N are subgroups in G such that $N \bigcap H = \{1\}$ and N is generated by a central element σ in G. We set $\mathcal{B} := \mathcal{A}^H$. We wish to prove that this setting fulfils the conditions in the preceding theorem, to get the following.

COROLLARY 3.18. *With the above notation, $\mathcal{B}[X, \sigma, 0] \subseteq \mathcal{A}[X, \sigma, 0]$ is a centrally $(\mathbb{K}H)^*$-Galois extensions.*

Proof. In order to apply Theorem 3.17, we have to check that σ is a morphism of $(\mathbb{K}H)^*$-comodules and that $\mathcal{B}^\sigma \subseteq \mathcal{A}^\sigma$ is a $(\mathbb{K}H)^*$-Galois extension. The former condition is equivalent to the fact that σ is a morphism of $\mathbb{K}H$-modules, which in our case means that $\sigma h = h\sigma$ for any h in H. Trivially this equality is satisfied as, by assumption, σ is central in G. Furthermore,

$$\mathcal{B}^\sigma = \left(\mathcal{A}^H\right)^\sigma = \left(\mathcal{A}^H\right)^N = \mathcal{A}^{HN} = \mathcal{K}.$$

A similar computation yields us $(\mathcal{A}^\sigma)^H = \mathcal{A}^{NH} = \mathcal{B}^\sigma$. On the other hand, since $N \bigcap H = \{1\}$ one can embed H into the group of field automorphisms of $\mathcal{A}^\sigma$ via the restriction map $u \mapsto u|_{\mathcal{A}^\sigma}$. By Artin's Lemma, $\mathcal{B}^\sigma \subseteq \mathcal{A}^\sigma$ is separable and normal of Galois group H. We have noticed in §2.34 that a finite field extension is $(\mathbb{K}H)^*$-Galois if and only if it is separable and normal of Galois group H. In conclusion, the second requirement is also satisfied. $\square$

References

[Ab]   E. Abe, *Hopf Algebras*, Cambridge Tracts in Mathematics **74**, Cambridge University Press, 1980.

[Br]   K. Brown, Cohomology of groups, Springer Verlag, Berlin, 1982.

[BŞ]   G. Bohm and D. Ştefan, *(Co)cyclic (Co)homology of Bialgebroids: An Approach via (Co)monads*, Commun. Math. Phys. **282** (2008), 239286.

[Bu]   D. Burghelea, *The cyclic homology of the group rings*, Comment. Math. Helv. **60** (1985), 354-365.

[CHR]   S.U. Chase, D.K. Harrison and A. Rosenberg, *Galois theory and cohomology of commutative rings*, Memoirs of the Amer. Math. Soc., No. 52, Amer. Math. Soc., Providence, 1965.

[CGW]   A. Căldăraru, A. Giaquinto and S. Witherspoon, *Algebraic deformations arising from orbifolds with discrete torsion*, J . Pure Appl. Algebra **187** (2004), No. 1-3, 51-70.

[CS]   S.U. Chase and M. Sweedler, *Hopf Algebras and Galois Theory*, Lecture Notes in Mathematics **97**, Springer Verlag, 1969.

[DeMI]   F. DeMeyer and E. Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Mathematics **181,** Springer Verlag, 1971.

[DNR]   S. Dăscălescu, C. Năstăsescu and Ş. Raianu, *Hopf algebras. An introduction,* Monographs and textbooks in pure and applied mathematics, no. **235**, M. Dekker, New York, 2001.

[D1]   Y. Doi, *Homological Coalgebra*, J. Math. Soc. Japan **33** (1981), 31-50.

[D2]   Y. Doi, *Algebras with total integrals*, Commun. Algebra **13** (1985), 2137-2159.

[FS]   M. Farinati and A. Solotar. *G-structure on the cohomology of Hopf algebras*, Proc. Am. Math. Soc. **132** (2004), No. 10, 2859-2865.

[FSS]   M. Farinati, A. Solotar and M. Suárez-Álvarez, *Hochschild homology and cohomology of generalized Weyl algebras*, Ann. Inst. Fourier **53**, No. 2, 465-488 (2003).

[HKRS]   P. M. Hajac, M. Khalkhali, B. Rangipour and Y. Sommerh¨auser *Hopf-cyclic homology and cohomology with coefficients*, C.R.Acad. Sci. Paris, Ser. I, Math. **338** (2004), 925–930.

[JŞ]   P. Jara and D. Ştefan, *Cyclic homology of Hopf algebras and Hopf Galois extensions*, Proc. Lond. Math. Soc. **93** (2006), No. 1, 138-174.

[Ka]   C. Kassel, *L'homologie cyclique des algèbres enveloppantes*, Inventiones Math. **91** (1988), 221-251.

[KT]   H.F. Kreimer and M. Takeuchi, *Hopf algebras and Galois extensions of an algebra*, Indiana Univ. Math. J. **30** (1981), 675−692.

[Lo1]   M. Lorenz, *On the homology of graded algebras*, Commun. Algebra **20** (1992), 489-507.

[Lo2]   M. Lorenz, *On Galois descent for Hochschild and cyclic homology*, Comment. Math. Helv. **69** (1994), No.3, 474-482.

[Mo]   S. Montgomery, *Hopf algebras and their actions on rings*, CMBS Regional conference series in mathematics, no. **82**, Providence, R.I., 1993.

[MW]   M. Mastnak and S. Witherspoon, *Bialgebra cohomology, pointed Hopf algebras, and deformations*, J. Pure Appl. Algebra **213** (2009), No. 7, 1399-1417.

[Nă]   C. Năstăsescu and F. Van Oystaeyen, *Graded Rings,* Springer Verlag, 2003.

[Ni]   V. Nistor, Group cohomology and the cyclic cohomology of crossed products, Inventiones Math. **99** (1990), 411-424.

[SS]   P. Schauenburg and H.J. Schneider, *On generalized Hopf Galois extensions*, J. Pure Appl. Algebra, **202** (2005), 168–194.

[Ş1]   D. Ştefan, *Hochschild cohomology of Hopf Galois extensions,* J. Pure Appl. Algebra **103** (1995), 221-233.

[Ş2]   D. Ştefan, *Decomposition of Hochschild cohomology* , Commun. Algebra, **24** (1996), 1695-1706.

[Wa]   W.C. Waterhouse, *Introduction to Affine Group Schemes* , Graduate text in Mathematics **66**, Springer-Verlag, 1979.

[We]   C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, 1994.

UNIVERSITÉ DE HAUTE ALSACE, LABORATOIRE DE MATHÉMATIQUES INFORMATIQUE ET APPLICATIONS, 4, RUE DES FRÈRES LUMIÈRE, 68093 MULHOUSE CEDEX, FRANCE
E–mail address: `Abdenacer.Makhlouf@uha.fr`

UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS, STR. ACADEMIEI 14, RO-70109, BUCHAREST, ROMANIA
E–mail address: `drgstf@gmail.com`